Decomposing graphs into edges and triangles*

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Abstract

We prove the following 30-year old conjecture of Győri and Tuza: the edges of every n-vertex graph G can be decomposed into complete graphs $C_1, \dots, C_\ell$ of orders two and three such that $|C_1| + \cdots + |C_\ell| \leq (1/2 + o(1))n^2$. This result implies the asymptotic version of the old result of Erdős, Goodman and Pósa that asserts the existence of such a decomposition with $\ell \leq n^2/4$.

1 Introduction

Results on the existence of edge-disjoint copies of specific subgraphs in graphs is one of the most classical themes in extremal graph theory. Motivated by the following result of Erdős, Goodman and Pósa [12], we study the problem of covering edges of a given graph by edge-disjoint complete graphs.

Theorem 1 (Erdős, Goodman and Pósa [12]). *The edges of every n-vertex graph can be decomposed into at most $\lfloor n^2/4 \rfloor$ complete graphs.*

In fact, they proved the following stronger statement.

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Theorem 2 (Erdős, Goodman and Pósa [12]). *The edges of every n-vertex graph can be decomposed into at most $\lfloor n^2/4 \rfloor$ copies of K_2 and K_3.*

The bounds given in Theorems 1 and 2 are best possible as witnessed by complete bipartite graphs with parts of equal sizes.

Theorem 1 actually holds in a stronger form that we now present. Chung [8], Győri and Kostochka [20], and Kahn [24], independently, proved a conjecture of Katona and Tarján asserting that the edges of every n-vertex graphs can be covered with complete graphs $C_1, \ldots, C_\ell$ such that the sum of their orders is at most $n^2/2$. In fact, the first two proofs yield a stronger statement, which implies Theorem 1 and which we next state as a separate theorem. To state the theorem, we define $\pi_k(G)$ for a graph G to be the minimum integer m such that the edges of G can be decomposed into complete graphs $C_1, \ldots, C_\ell$ of order at most k with $|C_1| + \cdots + |C_\ell| = m$, and let $\pi(G) = \min_{k \in \mathbb{N}} \pi_k(G)$.

Theorem 3 (Chung [8]; Győri and Kostochka [20]). *Every n-vertex graph G satisfies that $\pi(G) \leq n^2/2$.*

Observe that Theorem 3 indeed implies the existence of a decomposition into at most $\lfloor n^2/4 \rfloor$ complete graphs. McGuinnes [31, 32] extended these results by showing that decompositions from Theorems 1 and 3 can be constructed in the greedy way, which confirmed a conjecture of Winkler of this being the case in the setting of Theorem 1.

In view of Theorem 2, it is natural to ask whether Theorem 3 holds under the additional assumption that all complete graphs in the decomposition are copies of K_2 and K_3, i.e., whether $\pi_3(G) \leq n^2/2$. Győri and Tuza [21] provided a partial answer by proving that $\pi_3(G) \leq 9n^2/16$ while conjecturing the following.

Conjecture 1 (Győri and Tuza [34, Problem 40]). *Every n-vertex graph G satisfies that $\pi_3(G) \leq (1/2 + o(1))n^2$.*

We remark that we state the conjecture in the version given by Győri in several of his talks and by Tuza in [34, Problem 40]; the paper [21] contains a version with a different lower term.

Our main result is the proof of Conjecture 1, which is given in Theorem 15. This also solves [34, Problem 41], which we state as Corollary 16. Our proof combines the flag algebra method, introduced by Razborov in [33], and the regularity method arguments. We use the flag algebra method to prove the fractional relaxation of Conjecture 1 and we then apply a blow-up lemma for edge-decompositions recently proven by Kim, Kühn, Osthus and Tyomkyn [26]. It should be noted that the fractional relaxation of Conjecture 1 itself would not yield Theorem 15 but the flag algebra arguments yield a stronger statement (Theorem 12), which can be combined with the blow-up lemma.

We would also like to mention a closely related variant of the problem suggested by Erdős, where the cliques in the decomposition have weights one less than their orders. Formally, define $\pi^-(G)$ for a graph to be the minimum m such that the edges of a graph G can be decomposed into complete graphs $C_1, \ldots, C_\ell$ with $(|C_1| - 1) + \cdots + (|C_\ell| - 1) = m$. The problem raised by Erdős asserts, see [34, Problem 43], that $\pi^-(G) \leq n^2/4$ for every n-vertex graph G. This problem remains open and was proven for K_4-free graphs only recently by Győri and Keszegh [18, 19], who proved that every K_4-free graph with n vertices and $\lfloor n^2/4 \rfloor + k$ edges contains k edge-disjoint triangles.

2 Preliminaries

We follow the standard graph theory terminology; we review here some less standard notation and introduce the terminology related to design theory, the flag algebra method and the regularity method. If G is a graph, then $|G|$ denotes the number of vertices of G and $||G||$ the number of edges. Further, if W is a subset of vertices of G, then $G[W]$ is the subgraph of G induced by W, i.e., the subgraph with the vertex set W and all edges with end vertices inside W.

An (n, q, r, λ)-design is a collection $\mathcal{B}$ of q-element subsets of an n-element set such that every r-element subset is in exactly λ elements of $\mathcal{B}$. When λ is equal to one, the design is called a Steiner system. Designs do not exist for all choices of the parameters n, q, r and λ. In particular, the parameters must satisfy that $\binom{q-i}{r-i}$ divides $\lambda\binom{n-i}{r-i}$ for every $0 \leq i \leq r-1$. It was a long-standing open problem whether these necessary divisibility conditions are also sufficient for the existence of a design when n is large. The case where $r = 2$ was solved by Wilson in a series of papers [35–37] in the 1970's. However, the whole problem was settled only recently in a breakthrough paper by Keevash [25].

2.1 Flag algebra method

The flag algebra method introduced by Razborov [33] has changed the landscape of extremal combinatorics. It found its applications to many long-standing open problems, e.g. [2–7, 9–11, 13–17, 22, 23, 27–30]. The method is designed to analyze asymptotic behavior of substructure densities and we now briefly describe it.

We start with introducing necessary notation. The family of all finite graphs is denoted by $\mathcal{F}$ and the family of graphs with ℓ vertices by $\mathcal{F}_\ell$. If F and G are two graphs, then $p(F, G)$ is the probability that $|F|$ distinct vertices chosen uniformly at random among the vertices of G induce a graph isomorphic to F; if $|F| > |G|$, we set $p(F, G) = 0$. A type is a graph with its vertices labeled with $1, \ldots, |\sigma|$ and a σ-flag is a graph with $|\sigma|$ vertices labeled by $1, \ldots, |\sigma|$ such that the labeled vertices induce a copy of σ preserving the vertex labels. In the analogy with the notation for ordinary graphs, the set of all σ-flags is denoted by $\mathcal{F}^\sigma$ and the set of all σ-flags with exactly ℓ vertices by $\mathcal{F}^\sigma_\ell$.

We next extend the definition of $p(F, G)$ to σ-flags and generalize it to pairs of graphs. If F and G are two σ-flags, then $p(F,G)$ is the probability that $|F|-|\sigma|$ distinct vertices chosen uniformly at random among the unlabeled vertices of G induce a σ-flag F; if $|F|>|G|$, we again set $p(F,G)=0$. Let F and F' be two σ-flags and G a σ-flag with at least $|F|+|F'|-|\sigma|$ vertices. The quantity $p(F,F';G)$ is the probability that two disjoint $|F|-|\sigma|$ and $|F'|-|\sigma|$ subsets of unlabeled vertices of G induce together with the labeled vertices of G the σ-flags F and F', respectively. It holds [33, Lemma 2.3] that

$$p(F,F';G)=p(F,G)\cdot p(F',G)+o(1) \tag{1}$$

where $o(1)$ tends to zero with $|G|$ tending to infinity.

Let $\vec{F}=[F_1,\ldots,F_t]$ be a vector of σ-flags, i.e., $F_i\in\mathcal{F}^\sigma$. If M is a $t\times t$ positive semidefinite matrix, it follows from (1), see [33], that

$$0\le\sum_{i,j=1}^{t}M_{ij}p(F_i,G)p(F_j,G)=\sum_{i,j=1}^{t}M_{ij}p(F_i,F_j;G)+o(1). \tag{2}$$

The inequality (2) is usually applied to a large graph G with a randomly chosen labeled vertices in a way that we now describe. Fix σ-flags F and F' and a graph G. We now define a random variable $p(F,F';G^\sigma)$ as follows: label $|\sigma|$ vertices of G with $1,\ldots,|\sigma|$ and if the resulting graph G' is a σ-flag, then $p(F',F';G^\sigma)=p(F,F';G')$; if G' is not a σ-flag, then $p(F_i,F_j;G^\sigma)=0$. The expected value of $p(F,F';G^\sigma)$ can be expressed as a linear combination of densities of $(|F|+|F'|-|\sigma|)$-vertex subgraphs of G [33], i.e., there exist coefficients α_H, $H\in\mathcal{F}_{|F|+|F'|-|\sigma|}$, such that

$$\mathbb{E}\,p(F,F';G^\sigma)=\sum_{H\in\mathcal{F}_{|F|+|F'|-|\sigma|}}\alpha_H\cdot p(H,G) \tag{3}$$

for every graph G. It can be shown that $\alpha_H=\mathbb{E}\,p(F,F';H^\sigma)$.

Let $\vec{F}=[F_1,\ldots,F_t]$ be a vector of ℓ-vertex σ-flags and let M be a $t\times t$ positive semidefinite matrix. The equality (3) yields that there exist coefficients α_H such that

$$\mathbb{E}\sum_{i,j=1}^{t}M_{ij}p(F_i,F_j;G^\sigma)=\sum_{H\in\mathcal{F}_{2\ell-|\sigma|}}\alpha_H\cdot p(H,G) \tag{4}$$

for every graph G, which combines with (2) to

$$0\le\sum_{H\in\mathcal{F}_{2\ell-|\sigma|}}\alpha_H\cdot p(H,G)+o(1) \tag{5}$$

for every graph G, where

$$\alpha_H=\sum_{i,j=1}^{t}M_{ij}\cdot\mathbb{E}\,p(F_i,F_j;H^\sigma)$$

In particular, the coefficients α_H depend only on the choice of $\vec{F}$ and M.

2.2 Regularity method

In this subsection, we review the basic notions related to the Szemerédi Regularity Lemma and the blow-up lemma for edge-decompositions of Kim, Kühn, Osthus and Tyomkyn [26]. We start with presenting three definitions that we use further in our exposition.

Definition 4 (Density)**.** Let G be a graph and V and W two disjoint subsets of its vertices. The density of the pair (V, W) is equal to

$$d(V, W) := \frac{e(V, W)}{|V||W|},$$

where $e(V, W)$ is the number of edges between V and W.

Definition 5 (Regularity)**.** Let G be a graph, V and W two disjoint subsets of its vertices, and $\varepsilon \in (0, 1)$. We say that the pair (V, W) is ε-regular if the following holds for all subsets $V' \subset V$ and $W' \subset W$ with $|V'| \geq \varepsilon|V|$ and $|W'| \geq \varepsilon|W|$:

$$|d(V, W) - d(V', W')| \leq \varepsilon.$$

Definition 6 (Super-regularity)**.** Let G be a graph, V and W two disjoint subsets of its vertices, and $\varepsilon \in (0, 1)$. We say that the pair (V, W) is ε-super-regular if

- (V, W) is ε-regular,
- every vertex of V has at least $(d(V, W)-\varepsilon)|W|$ and at most $(d(V, W)+\varepsilon)|W|$ neighbors in W, and
- every vertex of W has at least $(d(V, W)-\varepsilon)|V|$ and at most $(d(V, W)+\varepsilon)|V|$ neighbors in V.

The Szemerédi Regularity Lemma reads as follows.

Lemma 7 (Regularity Lemma)**.** *For every real $\varepsilon > 0$ and integer $k_0 > 0$, there exists an integer K such that the vertices of every graph G with at least k_0 vertices can be partitioned into $k + 1$ subsets $V_0, \ldots, V_k$ where $k_0 \leq k \leq K$ such that*

- $|V_0| \leq \varepsilon|G|$,
- *the sets $V_1, \ldots, V_k$ have the same size, and*
- *all but at most εk^2 pairs (V_i, V_j) are ε-regular.*

Any partition $V_0, \ldots, V_k$ with the three properties given in Lemma 7 is called an ε-regular partition.

Definition 8 (Regularity graph)**.** Let G be a graph and $V_0, \ldots, V_k$ an ε-regular partition. The regularity graph R_G with respect to the partition $V_0, \ldots, V_k$ is the graph with k vertices such that the i-th and the j-th vertex, $1 \le i, j \le k$, are adjacent if and only if (V_i, V_j) is an ε-regular pair.

The following result was proven by Kim, Kühn, Osthus and Tyomkyn [26, Theorem 1.3]; we state the result in a version for non-spanning subgraphs, which is equivalent to the original statement.

Theorem 9. *For all $0 < d_0, \alpha_0 \le 1$ and $\Delta, r \in \mathbb{N}$ there exist $\varepsilon_0 > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n \ge n_0$. Let $H_1, \ldots, H_s$ be r-partite graphs such that each of them has r parts, each of size at most n, and its maximum degree is at most Δ. If G is an r-partite graph with parts of sizes n such that every pair of its parts is ε_0-super-regular with density at least d_0, and $||H_1|| + \cdots + ||H_s|| \le (1 - \alpha_0)||G||$, then G contains edge-disjoint copies of $H_1, \ldots, H_s$.*

The following proposition is a direct corollary of Theorem 9.

Proposition 10. *For every $\alpha \in (0, 1)$ and every $d \in (0, 1]$, there exists $\varepsilon > 0$ and $N \in \mathbb{N}$ with the following property. If G is a graph and V_1, V_2 and V_3 disjoint n-vertex subsets of its vertices, $n \ge N$, such that (V_i, V_j) is an ε-regular pair with density at least d for $1 \le i < j \le 3$, then G contains at least $dn^2 - \alpha n^2$ edge-disjoint triangles with one vertex in V_1, one in V_2 and one in V_3.*

Proof. Let $\varepsilon = \varepsilon_0/3$ and $N = \lceil n_0/(1 - 2\varepsilon) \rceil$, where ε_0 and n_0 are the values from Theorem 9 applied with $r = 3$, $\Delta = 2$, $d_0 = d/2$ and $\alpha_0 = \alpha/4$. We can assume that $\varepsilon \le \frac{\alpha}{8}$, $d - 4\varepsilon \ge d_0$ and $n_0 \ge 4/\alpha$.

For $i = 1, \ldots, 3$, let V_i' be the set of all vertices $v \in V_i$ such that v has at least $(d(V_i, V_j) - \varepsilon)|V_j|$ and at most $(d(V_i, V_j) + \varepsilon)|V_j|$ neighbors in V_j, $j \ne i$. Since all the pairs (V_i, V_j) are ε-regular, it follows that $|V_i'| \ge (1 - 2\varepsilon)|V_i|$. Let V_i'' be any $\lceil (1 - 2\varepsilon)n \rceil$-element subset of V_i'.

Let G' be the subgraph of G with the vertex set $V_1'' \cup V_2'' \cup V_3''$ and all edges between V_i'' and V_j'' with $i \ne j$. Note that every pair (V_i'', V_j'') is ε_0-super-regular with density at least $d - 4\varepsilon$. Set $H_i = K_3$, where $i = 1, \ldots, s$ and

$$s = \lceil (d - 4\varepsilon - \alpha/2)n^2 \rceil \le (d - 4\varepsilon - \alpha/2)n^2 + 1 \le (d - 4\varepsilon - \alpha_0)n^2 .$$

Theorem 9 now implies that G' has at least $s \ge (d - 4\varepsilon - \alpha/2)n^2 \ge (d - \alpha)n^2$ edge-disjoint triangles. □

3 Fractional decompositions

In this section, we prove a fractional relaxation of our main result. In order to apply our techniques to the original problem, we need to work in a more general setting of weighted graphs. A weighted graph G is a graph where each edge is assigned a weight between 0 and 1 (inclusively). For an integer k, we define a fractional k-decomposition of a graph G to be an assignment of non-negative real weights to complete subgraphs of order at most k such that the sum of the weights of complete subgraphs containing any edge e is equal to its weight. The weight of a k-decomposition is equal to the sum of the weights of complete subgraphs multiplied by their orders, and the minimum weight of a fractional k-decomposition of a weighted graph will be denoted by $\pi_{k,f}(G)$. Observe that if G is a graph with all edges of weight one, then $\pi_{k,f}(G) \leq \pi_k(G)$.

We start with proving the next lemma.

Lemma 11. *Let G be a weighted graph with all edges of weight one. It holds that*

$$\mathbb{E}_U \pi_{3,f}(G[U]) \leq 21 + o(1)$$

where U is a uniformly chosen random subset of seven vertices of G.

Proof. We use the flag algebra method to find coefficients c_U, $U \in \mathcal{F}_7$, such that

$$0 \leq \sum_{U \in \mathcal{F}_7} c_U \cdot p(U, G) + o(1) \tag{6}$$

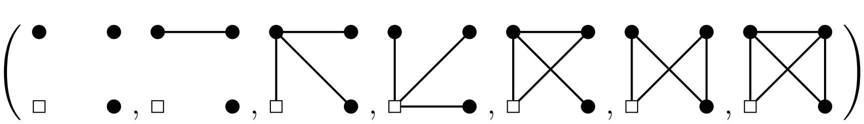
and

$$\pi_{3,f}(U) + c_U \leq 21 \tag{7}$$

for every $U \in \mathcal{F}_7$. The statement of the lemma would then follow from (6) and (7) using $\sum_{U \in \mathcal{F}_7} p(U, G) = 1$ as we next show.

$$\begin{aligned}\mathbb{E}_U \pi_{3,f}(G[U]) &= \sum_{U \in \mathcal{F}_7} \pi_{3,f}(U) \cdot p(U, G) \\ &\leq \sum_{U \in \mathcal{F}_7} (\pi_{3,f}(U) + c_U) \cdot p(U, G) + o(1) \\ &\leq \sum_{U \in \mathcal{F}_7} 21 \cdot p(U, G) + o(1) = 21 + o(1).\end{aligned}$$

We now focus on finding the coefficients c_U, $U \in \mathcal{F}_7$, satisfying (6) and (7). Let σ_1 be a flag consisting of a single vertex labeled with 1 and consider the following vector $\vec{F} = (F_1, \ldots, F_7)$ of σ_1-flags from $\mathcal{F}_4^{\sigma_1}$ (the single labeled vertex is depicted by a white square and the remaining vertices by black circles).

$\vec{F} =$ 

Let M be the following 7×7-matrix.

$$M = \frac{1}{12\cdot 10^9}\begin{pmatrix} 1800000000 & 2444365956 & 640188285 & -1524146769 & 1386815580 & -732139362 & -129387078 \\ 2444365956 & 4759879134 & 1177441152 & -1783771230 & 2546923788 & -1397639394 & -143552208 \\ 640188285 & 1177441152 & 484273772 & -317303211 & 1038156300 & -591902130 & -6783162 \\ -1524146769 & -1783771230 & -317303211 & 1558870290 & -651906630 & 305728704 & 154602378 \\ 1386815580 & 2546923788 & 1038156300 & -651906630 & 2285399634 & -1283125950 & -10755036 \\ -732139362 & -1397639394 & -591902130 & 305728704 & -1283125950 & 734039016 & -1621938 \\ -129387078 & -143552208 & -6783162 & 154602378 & -10755036 & -1621938 & 23860164 \end{pmatrix}.$$

The matrix M is a positive semidefinite matrix with rank six; the eigenvector corresponding to the zero eigenvalue is $(1,0,3,1,0,3,0)$. Let

$$c_U = \sum_{i,j=1}^{7} M_{ij}\mathbb{E}\, p(F_i, F_j; U^{\sigma_1})\ .$$

The inequality (5) implies that

$$0 \le \sum_{U\in\mathcal{F}_7} c_U \cdot p(U,G) + o(1),$$

which establishes (6). The inequality (7) is verified with computer assistance by evaluating the coefficient c_U and the quantity $\pi_{3,f}(U)$ for each $U \in \mathcal{F}_7$. Since $|\mathcal{F}_7| = 1044$, we we do not list c_U and $\pi_{3,f}(U)$ here. The computer programs that we used and their outputs have been made available on arXiv as ancillary files and are also available at `http://orion.math.iastate.edu/lidicky/pub/tile23`. □

We remark that if it were possible to prove Lemma 11 in the non-fractional setting, i.e., to show that $\mathbb{E}_U\pi_3(G[U]) \le 21 + o(1)$ using just the flag algebra method, one would not need to employ the regularity method tools to prove the main result of this paper. Unfortunately, the computation with 7-vertex flags yields only that $\mathbb{E}_U\pi_3(G[U]) \le 21.588 + o(1)$.

We now use Lemma 11 to prove the fractional relaxation of our main result.

Theorem 12. *Every n-vertex weighted graph G has a fractional 3-decomposition of weight at most $n^2/2 + o(n^2)$ such that each edge is contained in at most five triangles with positive weight.*

Proof. We can assume $\binom{7}{2}$ divides $\binom{n}{2}$ and 6 divides $n-1$ (if this were not the case, we would just add at most 42 isolated vertices to G, see [1] for further details on this constant). It follows that there exists $(n,7,2,1)$-design. Let m be the number of edges of G and $d_1,\dots,d_m$ their weights in the non-decreasing order; set $d_0 = 0$. Let G_i, $1 \le i \le m$, be the spanning unweighted subgraph of G formed exactly by the edges of weight at least d_i.

We construct a fractional 3-decomposition of G using the following random procedure. We first choose a $(n,7,2,1)$-design $\mathcal{B}$ uniformly at random among all $(n,7,2,1)$-designs on the vertex set of G; it follows that every 7-vertex subset is included in $\mathcal{B}$ with the same probability, which is equal to $\frac{n(n-1)}{42}\cdot\binom{n}{7}^{-1}$. Note that each pair of vertices of G is included in exactly one set contained in $\mathcal{B}$.

Fix an optimal fractional 3-decomposition of $G_i[B]$ for every subset B in $\mathcal{B}$ and every $i = 1, \ldots, m$. For every edge e of the graph G, we consider the unique subset of B containing both end vertices of e and define $w_i(e)$, $i = 1, \ldots, m$, to be the weight of e in the fractional 3-decomposition of $G_i[B]$ if the weight of e is at most d_i and to be zero otherwise. We next define weights $w_i(t)$ for each triangle t of the graph G. If there is a subset B in $\mathcal{B}$ containing all the three end vertices of t and the weights of all three edges of t are at most d_i, $i = 1, \ldots, m$, then $w_i(t)$ is the weight of t in the fractional 3-decomposition of $G_i[B]$. Otherwise, $w_i(t)$ is equal to zero.

We set the weight $w(e)$ of an edge e of G to be

$$w(e) = \sum_{i=1}^{m} (d_i - d_{i-1}) w_i(e)$$

and the weight $w(t)$ of a triangle of G to be

$$w(t) = \sum_{i=1}^{m} (d_i - d_{i-1}) w_i(t) \ .$$

The definition of the graphs G_i yield that w is a fractional 3-decomposition of G. Moreover, if $w(t) > 0$ for a triangle t of G, then all the three vertices of t lie in the common subset B in $\mathcal{B}$. In particular, each edge of G is contained in at most five triangles of positive weight.

We now show that the expected weight of the fractional 3-decomposition w is at most $n^2/2 + o(n^2)$. We use that every 7-vertex subset of vertices is included in $\mathcal{B}$ with the same probability, which implies that

$$\mathbb{E} \sum_{e} w(e) + \mathbb{E} \sum_{t} w(t) = \sum_{i=1}^{m} (d_i - d_{i-1}) \frac{n(n-1)}{42} \, \mathbb{E}_U \pi_{3,f}(G[U]), \tag{8}$$

where U is a uniform random subset of seven vertices of G. We next use Lemma 11 to derive the following from (8).

$$\begin{aligned}
\mathbb{E} \sum_{e} w(e) + \mathbb{E} \sum_{t} w(t) &\leq \sum_{i=1}^{m} (d_i - d_{i-1}) \frac{n(n-1)}{42} (21 + o(1)) \\
&= \sum_{i=1}^{m} (d_i - d_{i-1}) \frac{n^2}{2} + o(n^2) \\
&= (d_m - d_0) \left(\frac{n^2}{2} + o(n^2) \right) \leq \frac{n^2}{2} + o(n^2) \ .
\end{aligned}$$

Hence, the expected weight of the fractional 3-decomposition w is at most $n^2/2 + o(n^2)$. $\square$

4 Main result

We start with proving the following auxiliary lemma; its proof is a simple application of the probabilistic method and we include it for completeness.

Lemma 13. *For every integer $r \in \mathbb{N}$ and reals $\varepsilon \in (0, 1/4)$ and $\delta \in (0, 1)$, there exists n_0 such that the following holds. For every graph G, every ε-regular pair (V, W) of vertices of G with $|V| = |W| \geq n_0$, and all non-negative reals $d_1, \ldots, d_r$ such that $d_1 + \cdots + d_r \leq d(V, W)$, there exists a partition $E_1, \ldots, E_r$ of the edges between V and W such that the pair (V, W) when restricted to the edges in E_i, $i = 1, \ldots, r$, is an 3ε-regular with density at least $d_i - \delta$.*

We use the Chernoff Bound to prove the lemma, which we now state for reference.

Proposition 14 (Chernoff Bound)**.** *Let X be the sum of n independent random zero-one variables, each being one with probability p. It holds*

$$\mathbb{P}[|X - pn| \geq a] < 2e^{-\frac{a^2}{3pn}}$$

for every real $a \in \mathbb{R}$.

We are now ready to prove Lemma 13.

Proof of Lemma 13. Fix r, ε and δ, and consider a graph G together with an ε-regular pair (V, W) and reals $d_1, \ldots, d_r$ as in the statement of the lemma. We can assume without loss of generality that $d_1 + \cdots + d_r = d(V, W)$ and that $\delta \leq \varepsilon$. Also let $n = |V| = |W|$.

We randomly partition the edges between V and W into sets $E_1, \ldots, E_r$ in such a way that each edge is included in E_i with probability $p_i = \frac{d_i}{d(V,W)}$ independently of the other edges. The probability that E_i contains fewer than $(d_i - \delta)n^2$ edges or more than $(d_i + \delta)n^2$ edges is at most

$$2e^{-\frac{\delta^2 n^4}{3p_i n^2}} \leq 2e^{-\frac{\delta^2 n^2}{3}} \tag{9}$$

by Proposition 14. Next consider subsets $V' \subseteq V$ and $W' \subseteq W$ with $|V'|, |W'| \geq 3\varepsilon n$. The probability that the density of the pair (V', W') restricted to E_i differs from $p_i d(V', W')$ by more than ε is at most

$$2e^{-\frac{\varepsilon^2 |V'|^2 |W'|^2}{3p_i d(V',W')|V'||W'|}} \leq 2e^{-\frac{\varepsilon^2 |V'||W'|}{3}} \leq 2e^{-3\varepsilon^4 n^2} \tag{10}$$

by Proposition 14. Since the pair (V, W) is ε-regular, it holds that $|d(V, W) - d(V', W')| \leq \varepsilon$. It follows that the probability that the density of the pair (V', W') restricted to E_i differs from d_i by more than 2ε is at most $2e^{-3\varepsilon^4 n^2}$. The union bound applied with the estimate (10) yields that the probability that there exist such subsets V' and W' for some i is at most

$$r \cdot 2^{2n+1} \cdot e^{-3\varepsilon^3 n^2} . \tag{11}$$

We now choose n_0 such that each of the estimates (9) and (11) is at most $1/2r$ for every $n \geq n_0$. Hence, there is a positive probability that every E_i, $i = 1, \ldots, r$, contains between $(d_i - \delta)n^2$ and $(d_i + \delta)n^2$ edges (inclusively), i.e., the density of (V, W) restricted to E_i is between $d_i - \delta$ and $d_i + \delta$, and that all subsets $V' \subseteq V$ and $W' \subseteq W$, $|V'|, |W'| \geq 3\varepsilon n$, satisfy that the density of the pair (V', W') restricted to E_i differs from d_i by at most 2ε. Since such a partition satisfies that the pair (V, W) restricted to E_i is 3ε-regular (we use that $\delta \leq \varepsilon$) for every $i = 1, \ldots, r$, the statement of the lemma follows. □

We are now ready to prove the main result of the paper.

Theorem 15. *Every n-vertex graph G satisfies that $\pi_3(G) \leq (1/2 + o(1))n^2$.*

Proof. We show that for every $\delta > 0$, there exists N such that $\pi_3(G) \leq n^2/2 + \delta n^2$ for every graph G with $n \geq N$ vertices. Fix $\delta > 0$. We can assume without loss of generality that δ^{-1} is an integer.

Let ε_a and N_a be the values of ε and N from Proposition 10 applied for $\alpha = \delta/20$ and $d = a\delta/20$ where $a = 1, \ldots, 20\delta^{-1}$. Next set

$$\varepsilon = \min\{\delta/20, \varepsilon_1/3, \ldots, \varepsilon_{20\delta^{-1}}/3\} .$$

Let n_f be such that the $o(n^2)$ term in Theorem 12 is at most $\delta n^2/20$ for all $n \geq n_f$. We apply the Szemerédi Regularity Lemma (Lemma 7) with ε and $k_0 = \max\{20\delta^{-1}, n_f\}$ to get an integer K and Lemma 13 with $r = 6$, ε and $\delta/20$ to get an integer n_0, and set N to be any integer larger than $n_0 K(1-\varepsilon)^{-1}$ and larger than $N_a K(1-\varepsilon)^{-1}$ for $a = 1, \ldots, 20\delta^{-1}$.

Let G be a graph with $n \geq N$ vertices. By the Szemerédi Regularity Lemma, there exists an ε-regular partition $V_0, \ldots, V_k$ of the vertex set of G, where $k_0 \leq k \leq K$. Let R_G be the regularity graph with respect to the partition $V_0, \ldots, V_k$ and let v_i be the vertex of R_G corresponding to the part V_i, $i = 1, \ldots, k$. If (V_i, V_j) is ε-regular, assign the edge joining $v_i v_j$ the weight equal to $d(V_i, V_j)$.

By Theorem 12, the graph R_G has a fractional 3-packing of total weight at most $k^2/2 + \delta k^2/20$ (since $k \geq n_f$). Fix such a fractional 3-packing, let $w(t)$ be the weight of a triangle t of R_G in the packing and $w(e)$ the weight of an edge e. Consider an edge $v_i v_j$ of R_G. By Theorem 12, there are at most five triangles t containing $v_i v_j$ with $w(t) > 0$. Lemma 13 yields that there exist disjoint subsets E_{ij}^t of the edges between V_i and V_j, where t ranges through the at most five triangles containing $v_i v_j$ with $w(t) > 0$, such that E_{ij}^t contains at least $(w(t) - \delta/20)|V_i||V_j|$ edges and the pair (V_i, V_j) restricted to E_{ij}^t is 3ε-regular. Fix such sets E_{ij}^t for all ε-regular pairs (V_i, V_j).

Let n_V be the number of vertices contained in each of the parts $V_1, \ldots, V_k$; note that $n_V \geq n_0$ by the choice of N. For every triangle $t = v_i v_{i'} v_{i''}$ with $w(t) > 0$, we construct a large family of edge-disjoint triangles with edges from $E^t_{ii'}$, $E^t_{ii''}$ and $E^t_{i'i''}$. Let a be the largest integer such that $w(t) \geq (a+1)\delta/20$. Note that $n_V \geq N_a$ and that each of the sets $E^t_{ii'}$, $E^t_{ii''}$ and $E^t_{i'i''}$ has density at least $a\delta/20$ between the corresponding vertex parts. We apply Proposition 10 for the sets V_i, $V_{i'}$ and $V_{i''}$ with edges from $E^t_{ii'}$, $E^t_{ii''}$ and $E^t_{i'i''}$ and with $\alpha = \delta/20$ and $d = a\delta/20$. This yields a family of at least $dn_V^2 - \alpha n_V^2 \geq (w(t) - \delta/10)n_V^2$ edge-disjoint triangles with edges from $E^t_{ii'}$, $E^t_{ii''}$ and $E^t_{i'i''}$. Consider such a family of at least $(w(t) - \delta/10)n_V^2$ and at most $w(t)n_V^2$ triangles for each triangle t with $w(t) > 0$ and let $\mathcal{T}$ be the union of all such families for t with $w(t) > 0$. Note that the number of triangles contained in $\mathcal{T}$ is at most

$$\sum_t w(t)n_V^2 \leq \frac{n^2}{k^2} \sum_t w(t) \; . \tag{12}$$

Since each edge $v_i v_j$ of R_G is contained in at most five triangles with positive weight, we obtain that if (V_i, V_j) is an ε-regular pair, then the triangles contained in $\mathcal{T}$ cover all but at most $(w(v_i, v_j) + \delta/2)n_V^2$ edges between V_i and V_j.

We next estimate the number of edges that are not between (V_i, V_j) forming an ε-regular pair. There are three kinds of such edges: those incident with a vertex from V_0, those with both end vertices inside V_i for some $i = 1, \ldots, k$ and those between parts V_i and V_j, $1 \leq i < j \leq k$, such that (V_i, V_j) is not ε-regular. The number of edges incident with a vertex from V_0 is at most

$$|V_0|n \leq \varepsilon n^2 \leq \delta n^2/20 \; . \tag{13}$$

The number of edges with both end vertices inside the same part V_i for some $i = 1, \ldots, k$ is at most

$$k\binom{n_V}{2} \leq \frac{n^2}{2k} \leq \frac{n^2}{2k_0} \leq \delta n^2/40 \; . \tag{14}$$

Finally, the number of edges between parts V_i and V_j, $1 \leq i < j \leq k$, such that (V_i, V_j) is not ε-regular is at most

$$\varepsilon k^2 n_V^2 \leq \varepsilon n^2 \leq \delta n^2/20 \; . \tag{15}$$

Using (13), (14) and (15), we conclude that the number of edges not contained in a triangle in $\mathcal{T}$ is at most

$$\begin{aligned} \frac{5\delta n^2}{40} + \sum_e (w(v_i, v_j) + \delta/2)n_V^2 &\leq \frac{\delta n^2}{8} + \frac{\delta n^2}{4} + \frac{n^2}{k^2}\sum_e w(v_i, v_j) \\ &= \frac{3\delta n^2}{8} + \frac{n^2}{k^2}\sum_e w(v_i, v_j) \; . \end{aligned} \tag{16}$$

Since the total weight of the fractional 3-packing of R_G is at most $k^2/2 + \delta k^2/20$, we get from (12) and (16) that the triangles from $\mathcal{T}$ and the edges not covered by $\mathcal{T}$ (viewed as complete graphs of order two) form a 3-packing of G of total weight at most

$$\frac{3\delta n^2}{4} + \frac{n^2}{k^2}\left(\sum_e 2w(v_i,v_j) + \sum_t 3w(t)\right) \le \frac{3\delta n^2}{4} + \frac{n^2}{k^2}\left(\frac{k^2}{2} + \delta\frac{k^2}{20}\right) \le \frac{n^2}{2} + \delta n^2 \ .$$

The proof of the theorem is now finished. □

The next corollary follows directly from Theorem 15.

Corollary 16. *Every n-vertex graph with $n^2/4 + k$ edges contains $2k/3 - o(n^2)$ edge-disjoint triangles.*

5 Conclusion

We would like to conclude with mentioning two open problems related to our main result. Theorem 15 asserts that $\pi_3(G) \le n^2/2 + o(n^2)$ for every n-vertex graph G. However, it could be true (cf. the remark after Problem 41 in [34]) that $\pi_3(G) \le n^2/2 + 2$ for every n-vertex graph G. The second problem that we would like to mention is a possible generalization of Corollary 16, which is stated in [34] as Problem 42. Fix $r \ge 4$. Does every n-vertex graph with $\frac{r-2}{2r-2}n^2 + k$ edges contain $\frac{2}{r}k - o(n^2)$ edge-disjoint complete graphs of order k?

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