Journal of Combinatorial Theory, Series B 126 (2017) 83–113

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

Rainbow triangles in three-colored graphs

József Balogh [a,b,1], Ping Hu [c], Bernard Lidický [d,2], Florian Pfender [e,3], Jan Volec [f,g,4], Michael Young [d,5]

[a] *Department of Mathematics, University of Illinois, Urbana, IL 61801, USA*
[b] *Bolyai Institute, University of Szeged, Szeged, Hungary*
[c] *Department of Computer Science, University of Warwick, Coventry CV4 7AL, United Kingdom*
[d] *Department of Mathematics, Iowa State University, Ames, IA 50011, USA*
[e] *University of Colorado Denver, Mathematical and Statistical Sciences, Denver, CO, 80202, USA*
[f] *Department of Mathematics and Statistics, McGill University, 805 Sherbrooke West, Montreal H3A 2K6, Canada*
[g] *Mathematics Institute and DIMAP, University of Warwick, Coventry CV4 7AL, United Kingdom* [6]

ARTICLE INFO

Article history:
Received 12 August 2014
Available online 14 April 2017

Keywords:
Subgraph density
Flag algebra
Rainbow colorings

ABSTRACT

Erdős and Sós proposed the problem of determining the maximum number $F(n)$ of rainbow triangles in 3-edge-colored complete graphs on n vertices. They conjectured that $F(n) = F(a) + F(b) + F(c) + F(d) + abc + abd + acd + bcd$, where $a+b+c+d = n$ and a, b, c, d are as equal as possible. We prove that the conjectured recurrence holds for sufficiently large n. We also prove the conjecture for $n = 4^k$ for all $k \geq 0$. These

E-mail addresses: jobal@math.uiuc.edu (J. Balogh), p.hu@warwick.ac.uk (P. Hu), lidicky@iastate.edu (B. Lidický), florian.pfender@ucdenver.edu (F. Pfender), honza@ucw.cz (J. Volec), myoung@iastate.edu (M. Young).

[1] Research is partially supported by Simons Fellowship, NSF CAREER Grant DMS-0745185, Arnold O. Beckman Research Award (UIUC Campus Research Board 13039) and Marie Curie FP7-PEOPLE-2012-IIF 327763.

[2] Research partially done at University of Illinois, Urbana. Research of this author is supported in part by NSF grants DMS-1266016 and DMS-1600390.

[3] Research is partially supported by a grant from the Simons Foundation (# 276726) and by NSF grant DMS-1600483.

[4] Research is partially supported by the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ERC grant agreement no. 259385.

[5] Research supported in part by NSF grant DMS-0946431.

[6] Previous affiliation.

http://dx.doi.org/10.1016/j.jctb.2017.04.002
0095-8956/

results imply that $\lim \frac{F(n)}{\binom{n}{3}} = 0.4$, and determine the unique limit object. In the proof we use flag algebras combined with stability arguments.

1. Introduction

An edge-coloring of a graph (or a subgraph of a graph) is *rainbow* if each of its edges has a different color. Let G be a 3-edge-colored K_n, we define $F(G)$ to be the number of rainbow triangles in G, and define

$$F(n) = \max_{G\text{: 3-edge-colored } K_n} F(G).$$

The following conjecture on $F(n)$ was mentioned in [9] as an older problem of Erdős and Sós and it was mentioned again in [18].

Conjecture 1.

$$F(n) = F(a) + F(b) + F(c) + F(d) + abc + abd + acd + bcd, \tag{1}$$

where $a + b + c + d = n$ and a, b, c, d are as equal as possible.

This recursive formula arises from the following construction. Denote by Q a 3-edge-colored K_4, if it has the – up to isomorphism – unique coloring that every triangle in it is rainbow.

Construction 2. Fix an Q, and blow up its four vertices into four classes, of sizes a, b, c, d. The edges between two classes should inherit the color of the edge from the starting Q. This way, each of the triangles having vertices in three different classes will be rainbow. Inside of each class place an extremal coloring of K_a, K_b, K_c, K_d, see Fig. 1.

A slight strengthening of Conjecture 1 is as follows.

Conjecture 3. *For every n, all 3-colorings of K_n attaining $F(n)$ are attained via Construction 2.*

Up to a permutation of the colors in each iterative step, this construction gives a unique candidate for an extremal 3-coloring of all edges of K_n. Note that for $n = 4^k$, the allowed color permutations in each step are in fact isomorphisms, so in this case the extremal coloring is conjectured to be unique up to isomorphism. In this paper, we prove Conjecture 3 for large enough n, and for $n = 4^k$ for all $k \geq 0$.

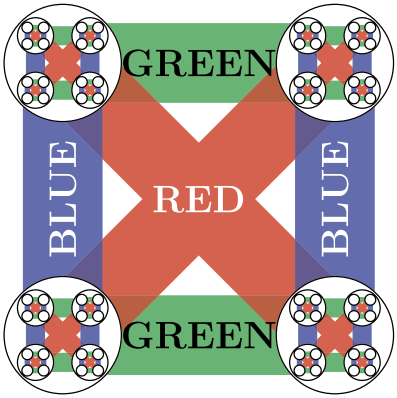

Fig. 1. Sketch of conjectured extremal construction $G^{\square}$.

Theorem 4. *There exists n_0 such that for every $n > n_0$*

$$F(n) = F(a) + F(b) + F(c) + F(d) + abc + abd + acd + bcd, \tag{2}$$

where $a + b + c + d = n$ and a, b, c, d are as equal as possible.

Moreover, if G is a 3-edge-colored complete graph on n vertices containing $F(n)$ rainbow triangles, then $V(G)$ can be partitioned into four sets X_1, X_2, X_3 and X_4 of sizes a, b, c and d respectively, such that the edges containing vertices from different classes are colored as in a blow-up of a properly 3-edge-colored K_4, where vertices of the K_4 are blown-up by a, b, c and d vertices.

Theorem 5. *Conjecture 3 holds for $n = 4^k$, where $k \geq 1$. Moreover, the unique extremal example is the $(k-1)$-times iterated blow-up of* Q.

We are not able to prove Conjecture 3 for all smaller n which are not powers of 4. Nevertheless, Theorem 4 is strong enough to directly imply the uniqueness of the extremal limit homomorphism (in the flag algebra sense), and thus the asymptotic density of rainbow triangles.

Theorem 6. *The unique limit homomorphism maximizing the density of rainbow triangles is given by the sequence of the iterated blow-ups of* Q. *This implies that*

$$\lim_{n\to\infty} \frac{F(n)}{\binom{n}{3}} = 0.4.$$

Counting the number of rainbow copies of given subgraphs was studied earlier, see for example [4] on a similar problem on hypercubes. Another natural question about triangles in 3-colored complete graphs, determining the *minimum* number of the monochromatic triangles, was solved in [8].

One of the tools we use to prove Theorem 6 is the method of flag algebras. The method was introduced by Razborov [21] as a general tool to approach problems from extremal combinatorics. The method has been successfully applied to various problems in extremal combinatorics. To name some of the applications, they were used for attacking the Caccetta–Häggkvist conjecture [15,24], Turán-type problems in graphs [13,14,19,20,22,25,26], 3-graphs [2,10,12] and hypercubes [1,5], extremal problems in a colored environment [3,8], and also to problems in geometry [17] or extremal theory of permutations [7]. For more details on these applications, see a recent survey of Razborov [23].

In the case when flag algebras give a sharp bound on the density, usually the extremal structure is 'clean'. Even then, to obtain an exact result, it requires obtaining extra information from the flag algebra computations, and then applying some stability type method. In most cases, this last step uses results from the computation that certain small substructures appear with density $o(1)$.

For our problem, the conjectured extremal structure has an iterated structure, for which it is quite rare to obtain the precise density from flag algebra computations alone, see for example the problem on inducibility of small out-stars in oriented graphs [10] (note that the problem of inducibility of all out-stars was recently solved by Huang [16] using different techniques). In our case, a direct application of the semidefinite method gives only an upper bound on the limit value and shows that $\lim_{n\to\infty} \frac{F(n)}{\binom{n}{3}} < 0.40005$. However, using flag algebras to find bounds on densities of other substructures and combining them with other combinatorial arguments, we manage to obtain the precise result, at least when n is a power of 4, or when n is sufficiently large.

The method introduced was successfully applied to the problem of maximization of induced 5-cycles in [6]. We hope that our methods may give some insights on how to attack some other hard problems.

2. Notation

Given a graph G, we use $V(G)$ and $E(G)$ to denote its vertex set and edge set respectively, and write $v(G) = |V(G)|$. We say that a 3-edge-colored complete graph G on n vertices is *extremal* if G contains the maximum number of rainbow triangles among all 3-edge-colored complete graphs on n vertices.

Notice that the three colors considered in our problem are interchangeable. To reduce the complexity of the computations, we will exploit this symmetry and consider 3-edge-*partitioned* graphs instead of 3-edge-colored graphs in Section 4. A complete graph is 3-edge-*partitioned* if its edge set is partitioned into at most three unlabeled sets. Given two 3-edge-partitioned complete graphs G and G', an isomorphism between G and G' is a bijection $f : V(G) \to V(G')$ such that every pair of edges $\{u_1, u_2\}, \{v_1, v_2\} \in E(G)$ is in the same part of $E(G)$ if and only if $\{f(u_1), f(u_2)\}, \{f(v_1), f(v_2)\} \in E(G')$ are in the same part of $E(G')$. Two 3-edge-partitioned complete graphs G and G' are *isomorphic*, which we denote by $G \cong G'$, if and only if there exists an isomorphism between G and G'.

In this sense, 3-edge-partitioning is a relaxation of 3-edge-coloring. In our problem, the two concepts are equivalent, as a rainbow triangle in a 3-edge-colored graph corresponds to a triangle containing an edge from each part in a 3-edge-partitioned graph. All theorems above have equivalent statements in terms of edge-partitions. You can obtain an extremal graph in one setting by labeling/unlabeling the parts of the edge partition in an extremal graph in the other setting.

For a 3-edge-partitioned complete graph G and a vertex set $U \subseteq V(G)$, denote by $G[U]$ the induced 3-edge-partitioned subgraph of G by the vertex set U. For a vertex v of G, we abbreviate $G[V \setminus \{v\}]$ to $G - v$.

Let H be a 3-edge-partitioned complete graph on t vertices and G be a 3-edge-colored complete graph on n vertices with $n \geq t$. Denote by $P(H, G)$ the number of t-subsets U of $V(G)$ such that $G[U] \cong H$, and define the *density* of H in G to be

$$p(H,G) = \frac{P(H,G)}{\binom{n}{t}}.$$

In other words, $p(H, G)$ is the probability that a random subset of $V(G)$ of size t induces a copy of H.

Fix a 3-edge-partitioned complete graph G. We denote by RBT the density of triangles with an edge in each of the three parts, i.e., the probability that random 3 vertices from G span edges from all three parts (rainbow triangles). Analogously, let TCT be the probability that random 3 vertices from G span edges from exactly two parts (two-colored triangles), and MONOT the probability that random 3 vertices from G span edges from only one part (monochromatic triangles). Note that both TCT and MONOT can be expressed as a linear combination of subgraph densities (in fact, each of them can be expressed as a combination of three subgraph densities). Also note that $\mathrm{RBT} + \mathrm{TCT} + \mathrm{MONOT} = 1$.

By Q (for quadrangle), we denote the density of K_4s, where the edges in each part form a matching on two edges. Let Q_2 be the probability that a random set of 5 vertices from G induces a 3-edge-partitioned graph containing exactly two copies of Q. In other words, the vertices induce a 5-vertex blow-up of Q, where the edge inside the unique blob of size 2 can be in any part. Next, we write Q^+ for the probability that a random set of 5 vertices from G contains exactly one copy of Q. Again, the values of Q_2 and Q^+ can be expressed as a linear combination of subgraph densities, and it follows that $\mathrm{Q} = 2/5 \cdot \mathrm{Q}_2 + 1/5 \cdot \mathrm{Q}^+$.

Finally, we define Q_3 and Q_{22} to be the probabilities that a random set of 6 vertices from G induces the appropriate 6-vertex blow-up of Q. Specifically, Q_3 is the probability that the induced graph is obtained from Q by blowing-up one of its vertices twice and assigning the three edges inside the blob arbitrarily to the parts. By Q_{22} we denote the other option – the probability that we choose two different vertices of Q and blow-up both of them once. See Fig. 2 for examples. As in all the previous cases, both Q_3 and Q_{22} can be expressed as an appropriate linear combination of subgraph densities.

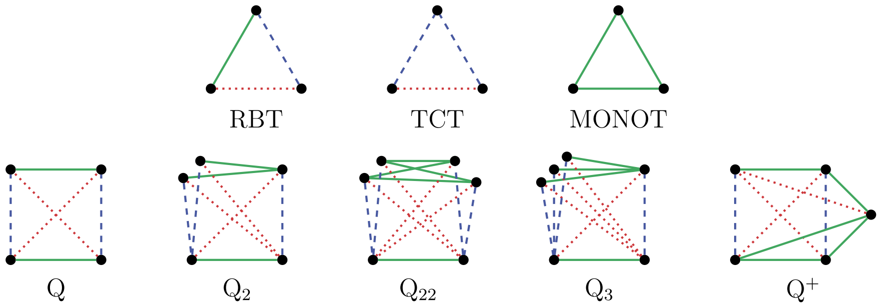

Fig. 2. Examples of small configurations.

Hence we call any of the probabilities defined in the last three paragraphs a *density expression*. In fact, these expressions are very closely related to the theory of flag algebras which we will discuss in Section 4. With a slight abuse of notation, we will also use the same notation for the corresponding classes of subgraphs and their members.

Let G be an extremal graph on n vertices and let D be some density expression. For any $X \subseteq V(G)$, we denote by $D(X)$ the density expression D restricted to subgraphs of G containing X, and we call $D(X)$ the *rooted density expression* of D at X in G. For example, for $X = \{x_1, x_2, x_3, x_4\}$, the rooted density expression $\mathrm{Q}_{22}(X)$ is the probability that random $6 - |X| = 2$ vertices from $V(G) \setminus X$ extends X to a subgraph from Q_{22}. Equivalently, it is the number of Q_{22}s containing the four vertices x_1, x_2, x_3, x_4 divided by $\binom{n-4}{2}$. For a fixed vertex $u \in V(G)$, we write $D(u)$ instead of $D(\{u\})$. Similarly, for a fixed edge vw, we write $D(vw)$ instead of $D(\{v, w\})$.

3. Outline of the proof of Theorem 4

The proof has some technical parts, so we give a thorough outline of the main ideas and motivations. Theorems 5 and 6 are consequences of Theorem 4, which we will prove in Section 5. Note that the first statement in Theorem 4 is a direct consequence of the second statement, so we only need to show the later one. We assume that G is a 3-edge-partitioned graph on n vertices maximizing the number of rainbow triangles.

Our first goal is to show that the vertices of G can be partitioned into four sets X_1, X_2, X_3, X_4 of almost equal size such that almost all edges between the sets are partitioned as in a blow-up of Q, see Fig. 3. In this quest, we are guided by what would happen if Theorem 4 is true. In this case, the vast majority of graphs $Z \in \mathrm{Q}$ contain exactly one vertex in each of the X_i. Further, adding any one vertex to such an *outer* Q yields a graph in Q_2, and adding two vertices yields a graph in $\mathrm{Q}_{22} \cup \mathrm{Q}_3$. Finally, we can in turn use such an outer $Z \in \mathrm{Q}$ to find the partition into the X_i, as each vertex is a twin (relative to Z) to exactly one of the four vertices of Z. In other words, given any $Z \in \mathrm{Q}$ we can check if it is outer and in this case recover the X_i by analyzing all the graphs we get by adding an extra vertex.

As we do not know at this point that Theorem 4 is true, and in particular we do not know the sets X_i in G, we start by carefully choosing a $Z \in \mathrm{Q}$ in G and use it to first partition the vertices of G into sets $Z_1, \dots, Z_4$ (which will eventually be modified to the X_i) and a trash set Z_0. In G, we call a $Z \in \mathrm{Q}$ *outer* if there are at least $n/2$ vertices v where $Z + v$ forms Q_2. Notice that the two definitions of outer Q are equivalent if Theorem 4 is true.

Following this idea, we want to pick Z in G, such that Z lies in many Q_2s, and determine the Z_i accordingly. We can find such a Z through an averaging argument from bounds given to us from some standard flag algebra computations: With the extra assumption that RBT ≥ 0.4, the computations give us bounds for Q and Q_2. But just knowing a bound on the number of Q_2s our graph Z lies in will not tell us anything about the relative sizes of the Z_i and about the partition of edges between sets, so this simple approach falls short of our goal.

To remedy this problem, we look at subgraphs of size 6 instead, and we pick Z to lie in many Q_{22}s. The bounds which we get from this show that the Z_i are somewhat balanced, most edges are in the right parts, and that Z_0 is small. But somewhat balanced is not strong enough for our later arguments to work. So we also bring Q_3 into the mix: Given Q_{22}, a smaller Q_3 means more balance. We look for a Z maximizing

$$\mathrm{Q}_{22}(Z) - \tfrac{26}{9}\mathrm{Q}_3(Z), \tag{3}$$

where the value $\frac{26}{9}$ comes from our attempt to minimize[7] the gap in (15) from Section 5. Again, the best we can do is to find a Z which achieves at least the average of (3) over all Q.

Unfortunately, the bounds on the Z_i we get from this Z are still not quite strong enough to later push through the whole proof, so we have to work yet a little harder. Notice that in the conjectured extremal graph $G^{\square}$, there are also Qs inside each of the four X_i. These *inner* Qs have much lower values in (3), so the average of that function is pushed down. On the other hand, if a vertex is added to an inner Q, in most cases it results in a copy of Q^+ and not Q_2 (which are always the result when starting from an outer Q). Following this observation, we consider the slightly bigger quantity

$$\mathrm{Q}_{22}(Z) - \tfrac{26}{9}\mathrm{Q}_3(Z) + \tfrac{27}{1000}\mathrm{Q}^+(Z), \tag{4}$$

where again $\frac{27}{1000}$ comes from optimizing (15). We find a lower bound on the average of this quantity with flag algebra computations, which guarantees the existence of a suitable $Z \in \mathrm{Q}$, and we can guarantee that the resulting $\{Z_i\}_{i=1}^4$ are fairly balanced, and contain most vertices of G. An edge between Z_i and Z_j is *funky* for $1 \leq i < j \leq 4$ if it lies in a different part from what the Q spanned by Z suggests. There are only few funky edges, as every such edge reduces $\mathrm{Q}_{22}(Z)$. We remove (very few) vertices incident to too many

[7] If $\frac{26}{9}$ was replaced by 3, this function would be 0 if all classes have the same sizes, which would be useless for us. Using a number slightly smaller than 3 makes this quantity the most powerful.

funky edges from Z_i, and obtain $X_1, \ldots, X_4$ and a trash set X_0 of all remaining vertices, while still maintaining fairly strong bounds on the sizes of the X_is.

Using this structure, we can now step by step get closer to our goal. In Claim 12 we show that in the four subgraphs $G[X_i]$, no vertex can have a majority of its incident edges inside only one of the three parts. Otherwise, this vertex would lie in too few rainbow triangles, contradicting the simple Proposition 8 with the consequence that $\mathrm{RBT}(v) \geq 0.4 - o(1)$ for every vertex v in G.

The remainder of the proof uses mostly local edge-repartitioning arguments; we rule out certain scenarios by showing that moving some edges in these scenarios to another part would increase the number of rainbow triangles.

If some edge uv is funky with $v \in X_i$, then the vast majority of the edges from v to other vertices in X_i must be in the same part, as otherwise moving uv would increase the number of rainbow triangles. This is stated precisely in Claim 13.

The last two claims show that every vertex incident to funky edges must be incident to more than $0.4n$ edges of the same part. Using bounds from another flag algebra computation, we can show that this can occur only for very few vertices in Claim 15, and therefore the funky edges are incident to only a very small number of vertices. Using this knowledge, we can use an edge-repartitioning argument very similar to the one in Claim 13, yielding bounds contradicting Claim 12. This contradiction shows that in fact there are no funky edges.

Therefore for $1 \leq i < j \leq 4$, all the edges between X_i and X_j are in the right part, but we still need to deal with vertices in X_0. In Claims 17 and 18 we show that if we forcefully include a vertex from X_0 in any X_i, it will result in many funky edges. In other words, every vertex in X_0 looks very different from vertices in the other X_i. In fact, vertices in X_0 look so different from vertices in the X_i that we can show that they cannot lie in enough rainbow triangles, so X_0 must be empty. This last argument in Claim 19 relies on a massive case analysis handled by the computer, as we are maximizing a quadratic function over a 12-dimensional polytope with thousands of facets. To complete the proof, we show in Claim 20 that the sizes of the X_i are almost balanced.

4. Flag algebras

The aim of this section is to establish the following statement.

Proposition 7. *There exists $n_0 \in \mathbb{N}$ such that every extremal 3-edge-partitioned complete graph G on at least n_0 vertices has the following properties:*

$$\tfrac{4}{15}\mathrm{Q}_{22} - \tfrac{26}{45}\mathrm{Q}_3 + \tfrac{27}{5000}\mathrm{Q}^+ > 0.002629395; \tag{5}$$

$$\mathrm{RBT} < 0.40005; \tag{6}$$

$$\mathrm{Q} < 0.09523837; \tag{7}$$

$$\tfrac{1}{3}\mathrm{TCT} + \mathrm{MONOT} < 0.33343492. \tag{8}$$

Let us give the related subgraph densities in Construction 2:

$$\mathrm{Q}_{22} = 270/1023, \qquad \mathrm{Q}_3 = 120/1023,$$
$$\mathrm{Q}^+ = 2/357, \qquad \mathrm{RBT} = 0.4,$$
$$\mathrm{Q} = 2/21, \qquad \mathrm{TCT}/3 + \mathrm{MONOT} = 1/3.$$

We also list the arithmetic values of (5) to (8) for Construction 2 below:

$$\tfrac{4}{15}\mathrm{Q}_{22} - \tfrac{26}{45}\mathrm{Q}_3 + \tfrac{27}{5000}\mathrm{Q}^+ \approx 0.002636964;$$
$$\mathrm{RBT} = 0.4;$$
$$\mathrm{Q} \approx 0.095238095;$$
$$\tfrac{1}{3}\mathrm{TCT} + \mathrm{MONOT} \approx 0.333333333.$$

The main tool used for the proof of Proposition 7 is flag algebras.

4.1. Flag algebra terminology

Let us now introduce the terminology related to flag algebras needed in this paper. The central notions we are going to introduce are an algebra $\mathcal{A}$ and algebras $\mathcal{A}^\sigma$, where σ is a fixed 3-edge-partitioning of a complete graph.

In order to precisely describe the algebras $\mathcal{A}$ and $\mathcal{A}^\sigma$, we first need to introduce some additional notation. Let $\mathcal{F}$ be the set of all finite 3-edge-partitioned complete graphs. Next, for every $\ell \in \mathbb{N}$, let $\mathcal{F}_\ell \subset \mathcal{F}$ be the set of ℓ-vertex 3-edge-partitioned graphs from $\mathcal{F}$. For $H \in \mathcal{F}_\ell$ and $H' \in \mathcal{F}_{\ell'}$, recall that $p(H, H')$ is the probability that a randomly chosen subset of ℓ vertices in H' induces a subgraph isomorphic to H. Note that $p(H, H') = 0$ if $\ell' < \ell$. Let $\mathbb{R}\mathcal{F}$ be the set of all formal linear combinations of elements of $\mathcal{F}$ with real coefficients. Furthermore, let $\mathcal{K}$ be the linear subspace of $\mathbb{R}\mathcal{F}$ generated by all linear combinations of the form

$$H - \sum_{H' \in \mathcal{F}_{v(H)+1}} p(H, H') \cdot H'.$$

Finally, we define $\mathcal{A}$ to be the space $\mathbb{R}\mathcal{F}$ quotiented by $\mathcal{K}$.

The space $\mathcal{A}$ has naturally defined linear operations of addition, and a multiplication by a real number. We now introduce a multiplication inside $\mathcal{A}$. We first define it on the elements of $\mathcal{F}$ in the following way. For $H_1, H_2 \in \mathcal{F}$, and $H \in \mathcal{F}_{v(H_1)+v(H_2)}$, we define $p(H_1, H_2; H)$ to be the probability that a randomly chosen subset of $V(H)$ of size $v(H_1)$ and its complement induce in H subgraphs isomorphic to H_1 and H_2, respectively. We set

$$H_1 \times H_2 = \sum_{H \in \mathcal{F}_{v(H_1)+v(H_2)}} p(H_1, H_2; H) \cdot H.$$

The multiplication on $\mathcal{F}$ has a unique linear extension to $\mathbb{R}\mathcal{F}$, which yields a well-defined multiplication also in the factor algebra $\mathcal{A}$. A formal proof of this can be found in [21, Lemma 2.4].

Let us now move to the definition of an algebra $\mathcal{A}^\sigma$, where $\sigma \in \mathcal{F}$ is an arbitrary 3-edge-partitioned complete graph with a fixed labeling of its vertex set. The labeled graph σ is usually called a *type* within the flag algebra framework. Without loss of generality, we will assume that the vertices of σ are labeled by $1, 2, \ldots, v(\sigma)$. Now we follow almost the same lines as in the definition of $\mathcal{A}$. We define $\mathcal{F}^\sigma$ to be the set of all finite 3-edge-partitioned complete graphs H with a fixed *embedding* of σ, i.e., an injective mapping θ from $V(\sigma)$ to $V(H)$ such that $\mathrm{im}(\theta)$ induces in H a subgraph isomorphic to σ. The elements of $\mathcal{F}^\sigma$ are usually called *σ-flags* and the subgraph induced by $\mathrm{im}(\theta)$ is called the *root* of a σ-flag.

Again, for every $\ell \in \mathbb{N}$, we define $\mathcal{F}^\sigma_\ell \subset \mathcal{F}^\sigma$ to be the set of the σ-flags from $\mathcal{F}^\sigma$ that have size ℓ (i.e., the σ-flags with the underlying 3-edge-partitioned graph having ℓ vertices). Analogously to the case for $\mathcal{A}$, for two 3-edge-partitioned graphs $H, H' \in \mathcal{F}^\sigma$ with the embeddings of σ given by θ, θ', we set $p(H, H')$ to be the probability that a randomly chosen subset of $v(H) - v(\sigma)$ vertices in $V(H') \setminus \theta'(V(\sigma))$ together with $\theta'(V(\sigma))$ induces a subgraph that is isomorphic to H through an isomorphism f that preserves the embedding of σ. In other words, the isomorphism f has to satisfy $f(\theta') = \theta$. Let $\mathbb{R}\mathcal{F}^\sigma$ be the set of all formal linear combinations of elements of $\mathcal{F}^\sigma$ with real coefficients, and let $\mathcal{K}^\sigma$ be the linear subspace of $\mathbb{R}\mathcal{F}^\sigma$ generated by all the linear combinations of the form

$$H - \sum_{H' \in \mathcal{F}^\sigma_{v(H)+1}} p(H, H') \cdot H'.$$

We define $\mathcal{A}^\sigma$ to be $\mathbb{R}\mathcal{F}^\sigma$ quotiented by $\mathcal{K}^\sigma$.

We now describe the multiplication of two elements from $\mathcal{F}^\sigma$. Let $H_1, H_2 \in \mathcal{F}^\sigma$, $H \in \mathcal{F}^\sigma_{v(H_1)+v(H_2)-v(\sigma)}$, and θ be the fixed embedding of σ in H. As in the definition of multiplication for $\mathcal{A}$, we define $p(H_1, H_2; H)$ to be the probability that a randomly chosen subset of $V(H) \setminus \theta(V(\sigma))$ of size $v(H_1) - v(\sigma)$ and its complement in $V(H) \setminus \theta(V(\sigma))$ of size $v(H_2) - v(\sigma)$, extend $\theta(V(\sigma))$ in H to subgraphs isomorphic to H_1 and H_2, respectively. This definition naturally extends to $\mathcal{A}^\sigma$.

Now consider an infinite sequence $(G_n)_{n\in\mathbb{N}}$ of 3-edge-partitioned complete graphs of increasing orders. We say that the sequence $(G_n)_{n\in\mathbb{N}}$ is *convergent* if the probability $p(H, G_n)$ has a limit for every $H \in \mathcal{F}$. A standard compactness argument (e.g., using Tychonoff's theorem) yields that every such infinite sequence has a convergent subsequence. All the following results can be found in [21]. Fix a convergent increasing sequence $(G_n)_{n\in\mathbb{N}}$ of 3-edge-partitioned graphs. For every $H \in \mathcal{F}$, we set $\phi(H) = \lim_{n\to\infty} p(H, G_n)$ and linearly extend ϕ to $\mathcal{A}$. We usually refer to the mapping ϕ as the limit of the sequence. The obtained mapping ϕ is a homomorphism from $\mathcal{A}$ to $\mathbb{R}$. Moreover, for every $H \in \mathcal{F}$, we obtain $\phi(H) \geq 0$. Let $\mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ be the set of all such homomorphisms, i.e., the set of all homomorphisms ψ from the algebra $\mathcal{A}$ to $\mathbb{R}$ such that $\psi(H) \geq 0$ for every $H \in \mathcal{F}$. It is interesting to see that this set is exactly the set of all limits of convergent sequences of 3-edge-partitioned complete graphs [21, Theorem 3.3].

Let $(G_n)_{n\in\mathbb{N}}$ be a convergent sequence of 3-edge-partitioned graphs and $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ be its limit. For $\sigma \in \mathcal{F}$ and an embedding θ of σ in G_n, we define G_n^θ to be the 3-edge-partitioned graph rooted on the copy of σ that corresponds to θ. For every $n \in \mathbb{N}$ and $H^\sigma \in \mathcal{F}^\sigma$, we define $p_n^\theta(H^\sigma) = p(H^\sigma, G_n^\sigma)$. Picking θ at random gives rise to a probability distribution $\mathbf{P}^\sigma_\mathbf{n}$ on mappings from $\mathcal{A}^\sigma$ to $\mathbb{R}$, for every $n \in \mathbb{N}$. Since $p(H, G_n)$ converges (as n tends to infinity) for every $H \in \mathcal{F}$, the sequence of these probability distributions on mappings from $\mathcal{A}^\sigma$ to $\mathbb{R}$ weakly converges [21, Theorems 3.12 and 3.13]. We denote the limit probability distribution by $\mathbf{P}^\sigma$. In fact, for any σ such that $\phi(\sigma) > 0$, the homomorphism ϕ itself fully determines the random distribution $\mathbf{P}^\sigma$ [21, Theorem 3.5]. Furthermore, any mapping ϕ^σ from the support of the distribution $\mathbf{P}^\sigma$ is in fact a homomorphism from $\mathcal{A}^\sigma$ to $\mathbb{R}$ such that $\phi^\sigma(H^\sigma) \geq 0$ for all $H^\sigma \in \mathcal{F}^\sigma$ [21, Proof of Theorem 3.5].

The last notion we introduce is the *averaging* (or downward) operator $[\![\cdot]\!]_\sigma : \mathcal{A}^\sigma \to \mathcal{A}$. It is a linear operator defined on the elements of $H^\sigma \in \mathcal{F}^\sigma$ by $[\![H^\sigma]\!]_\sigma = p_H^\sigma \cdot H^\emptyset$, where $H^\emptyset$ is the (unlabeled) 3-edge-partitioned graph from $\mathcal{F}$ corresponding to H^σ, and p_H^σ is the probability that a random injective mapping from $V(\sigma)$ to $V(H^\emptyset)$ is an embedding of σ in $H^\emptyset$ yielding a σ-flag isomorphic to H^σ. The key relation between ϕ and ϕ^σ is the following:

$$\forall H^\sigma \in \mathcal{A}^\sigma, \quad \phi\left([\![H^\sigma]\!]_\sigma\right) = \phi([\![\sigma]\!]_\sigma) \cdot \int \phi^\sigma(H^\sigma),$$

where the integration is over the probability space given by the random distribution $\mathbf{P}^\sigma$ on ϕ^σ. Therefore, if $\phi^\sigma(A^\sigma) \geq 0$ almost surely for some $A^\sigma \in \mathcal{A}^\sigma$, then $\phi\left([\![A^\sigma]\!]_\sigma\right) \geq 0$. In particular,

$$\forall A^\sigma \in \mathcal{A}^\sigma, \quad \phi\left(\left[\!\!\left[(A^\sigma)^2\right]\!\!\right]_\sigma\right) \geq 0. \tag{9}$$

The semidefinite method is a tool from the flag algebra framework that, for a given density problem of the form

$$\min_{\phi \in \mathrm{Hom}^+(\mathcal{A},\mathbb{R})} \phi(A),$$

where $A \in \mathcal{A}$, systematically searches for ‘best possible’ inequalities of the form (9). If we fix in advance an upper bound on the size of graphs in the terms of inequalities we will be using, we can find the best inequalities of the form (9) using semidefinite programming. Furthermore, it is easy to extend this basic semidefinite method in such a way that in addition to inequalities of the form (9), it uses a given list of other linear inequalities.

4.2. Proof of Proposition 7

We start this section by showing that in an extremal graph, every two vertices participate in almost the same number of rainbow triangles.

Proposition 8. *In an extremal graph G on n vertices, for any pair of vertices $u, v \in V(G)$, we have $\binom{n-1}{2}(\mathrm{RBT}(u) - \mathrm{RBT}(v)) \leq n-2$.*

Proof. Otherwise, we could delete v and duplicate u to u', i.e., for every vertex $x \neq u$ we could assign the edge xu' to the same part as xu. This implies that the part of the new edge uu' does not matter since uu' will not be in a rainbow triangle anyways. Let us call the new graph G'. Then

$$F(G') - F(G) \geq \binom{n-1}{2}(\mathrm{RBT}(u) - \mathrm{RBT}(v)) - \binom{n-2}{1}\mathrm{RBT}(uv)$$
$$\geq \binom{n-1}{2}(\mathrm{RBT}(u) - \mathrm{RBT}(v)) - (n-2) > 0,$$

a contradiction. □

Combining this with the bound given by Construction 2 yields the following.

Corollary 9. *In an extremal graph G, $\mathrm{RBT}(v) \geq 0.4 - o(1)$ for all vertices $v \in V(G)$.*

Let $(E_n)_{n\in\mathbb{N}}$ be any convergent sequence of extremal graphs of increasing orders with $e \in \mathrm{Hom}^+(\mathcal{F}, \mathbb{R})$ being its limit. We call such e an extremal limit. We now look at the additional properties that every extremal limit needs to satisfy. We start with a "flag algebra version" of Corollary 9.

Corollary 10. *Let σ be the 1-vertex type, RBT^σ be the σ-flag of size three with all three edges in different parts, e be an extremal limit and e^σ be a random homomorphism drawn from $\mathbf{P}^\sigma$ of e. Then with probability 1,*

$$e^\sigma\left(\mathrm{RBT}^\sigma - 1/4\right) \geq 0.$$

Furthermore, for any real $w \geq 0$ and $F^\sigma \in \mathcal{F}^\sigma$, it follows that

$$e\left(w \cdot [\![F^\sigma \times (\mathrm{RBT}^\sigma - 1/4)]\!]_\sigma\right) \geq 0. \tag{10}$$

Next, we apply four times the semidefinite method that seeks for inequalities of the form (9) and (10) to conclude the following.

Lemma 11. *For every extremal limit e:*

$$e\left(\tfrac{4}{15}\mathrm{Q}_{22} - \tfrac{26}{45}\mathrm{Q}_3 + \tfrac{27}{5000}\mathrm{Q}^+\right) \geq \frac{146593684097622593341208220713459404937779}{557518629963265578538392956816209037649510 4};$$

$$e\left(\mathrm{RBT}\right) \leq \frac{111516451991115812683901531193017407866460 69}{278759314981632789269196478408104518824755 20};$$

$$e\left(\mathrm{Q}\right) \leq \frac{265485807942351943716784898403205143897069}{278759314981632789269196478408104518824755 2};$$

$$e\left(\tfrac{1}{3}\mathrm{TCT} + \mathrm{MONOT}\right) \leq \frac{557688538928414953950562750058999625841387 7}{167255588988979673561517887044862711294853 12}.$$

Proof. At the beginning, we express all four left-hand sides as a linear combination of densities of 3-edge-partitioned complete graphs on 6 vertices. Note that $|\mathcal{F}_6| = 4300$.

The first inequality can be obtained as the sum of the following inequalities:

- 163 inequalities of the form $e\left(\left[\!\left[\left(\sum_{F\in\mathcal{F}_5^\sigma} x_F \cdot F\right)^2\right]\!\right]_\sigma\right) \geq 0$, where σ is a (not always the same) type on 4 vertices and $x_F \in \mathbb{Q}$ for all $F \in \mathcal{F}_5^\sigma$,
- 14 inequalities of the form $e\left(\left[\!\left[\left(\sum_{F\in\mathcal{F}_4^\sigma} x_F \cdot F\right)^2\right]\!\right]_\sigma\right) \geq 0$, where σ is the only 2-vertex type and $x_F \in \mathbb{Q}$ for all $F \in \mathcal{F}_4^\sigma$,
- one inequality of the form $e\left(\left(\sum_{F\in\mathcal{F}_3} x_F \cdot F\right)^2\right) \geq 0$, where $x_F \in \mathbb{Q}$ for all $F \in \mathcal{F}_3$,
- 17 inequalities of the form $e\left(w \cdot [\![F \times (\mathrm{RBT}^\sigma - 1/4)]\!]_\sigma\right) \geq 0$, where σ is the 1-vertex type, $w \geq 0$ and $F \in \mathcal{F}_4^\sigma$,
- an inequality of the form $e\left(\sum_{F\in\mathcal{F}_6} y_F \cdot F\right) \geq 0$, where $y_F \geq 0$ for all $F \in \mathcal{F}_6$,
- the equation $e\left(z \cdot \sum_{F_i\in\mathcal{F}_6} F_i\right) = z$, where $z = \frac{146593684097622593341208220713459404937779}{557518629963265578538392956816209037649510 4}$.

The second inequality can be obtained as the sum of the following inequalities:

- 884 inequalities of the form $e\left(-\left[\!\left[\left(\sum_{F\in\mathcal{F}_5^\sigma} x_F \cdot F\right)^2\right]\!\right]_\sigma\right) \leq 0$, where σ is a (not always the same) type on 4 vertices and $x_F \in \mathbb{Q}$ for all $F \in \mathcal{F}_5^\sigma$,
- 30 inequalities of the form $e\left(-\left[\!\left[\left(\sum_{F\in\mathcal{F}_4^\sigma} x_F \cdot F\right)^2\right]\!\right]_\sigma\right) \leq 0$, where σ is the only 2-vertex type and $x_F \in \mathbb{Q}$ for all $F \in \mathcal{F}_4^\sigma$,
- an inequality of the form $e\left(-\sum_{F\in\mathcal{F}_6} y_F \cdot F\right) \leq 0$, where $y_F \geq 0$ for all $F \in \mathcal{F}_6$,
- the equation $e\left(z \cdot \sum_{F_i\in\mathcal{F}_6} F_i\right) = z$, where $z = \frac{111516451991115812683901531193017407866460 69}{278759314981632789269196478408104518824755 20}$.

The third inequality can be obtained as the sum of the following inequalities:

- 948 inequalities of the form $e\left(-\left[\!\left[\left(\sum_{F\in\mathcal{F}_5^\sigma} x_F \cdot F\right)^2\right]\!\right]_\sigma\right) \leq 0$, where σ is a (not always the same) type on 4 vertices and $x_F \in \mathbb{Q}$ for all $F \in \mathcal{F}_5^\sigma$,
- 38 inequalities of the form $e\left(-\left[\!\left[\left(\sum_{F\in\mathcal{F}_4^\sigma} x_F \cdot F\right)^2\right]\!\right]_\sigma\right) \leq 0$, where σ is the only 2-vertex type and $x_F \in \mathbb{Q}$ for all $F \in \mathcal{F}_4^\sigma$,

- 15 inequalities of the form $e\left(-w \cdot [\![F \times (\mathrm{RBT}^\sigma - 1/4)]\!]_\sigma\right) \le 0$, where σ is the 1-vertex type, $w \ge 0$ and $F \in \mathcal{F}_4^\sigma$,
- an inequality of the form $e\left(-\sum_{F\in\mathcal{F}_6} y_F \cdot F\right) \le 0$, where $y_F \ge 0$ for all $F \in \mathcal{F}_6$,
- the equation $e\big(z \cdot \sum_{F_i\in\mathcal{F}_6} F_i\big) = z$, where $z = \frac{265485807942351943716784898403205143897069}{278759314981632789269196478408104518824 7552}$.

Finally, the last inequality can obtained as the sum of the following inequalities:

- 876 inequalities of the form $e\left(-\left[\!\!\left[\left(\sum_{F\in\mathcal{F}_5^\sigma} x_F \cdot F\right)^2\right]\!\!\right]_\sigma\right) \le 0$, where σ is a (not always the same) type on 4 vertices and $x_F \in \mathbb{Q}$ for all $F \in \mathcal{F}_5^\sigma$,
- 34 inequalities of the form $e\left(-\left[\!\!\left[\left(\sum_{F\in\mathcal{F}_4^\sigma} x_F \cdot F\right)^2\right]\!\!\right]_\sigma\right) \le 0$, where σ is the only 2-vertex type and $x_F \in \mathbb{Q}$ for all $F \in \mathcal{F}_4^\sigma$,
- 21 inequalities of the form $e\left(-w \cdot [\![F \times (\mathrm{RBT}^\sigma - 1/4)]\!]_\sigma\right) \le 0$, where σ is the 1-vertex type, $w \ge 0$ and $F \in \mathcal{F}_4^\sigma$,
- an inequality of the form $e\left(-\sum_{F\in\mathcal{F}_6} y_F \cdot F\right) \le 0$, where $y_F \ge 0$ for all $F \in \mathcal{F}_6$,
- the equation $e\big(z \cdot \sum_{F_i\in\mathcal{F}_6} F_i\big) = z$, where $z = \frac{557688538928414953950562750058999625841387 7}{1672555889889796735615178870448627112948531 2}$.

The exact rational values of all the coefficients x_F, y_F and w that appear in the inequalities above were obtained with computer assistance. They are available at http://www.math.uiuc.edu/~jobal/cikk/rbt, as well as a small Sage script that computes the corresponding sums. □

In order to prove Proposition 7, we just translate the previous statement back to the finite setting.

Proof of Proposition 7. Suppose one of the inequalities from the statement of Proposition 7 is false. For example, suppose that the inequality (6) is false. Therefore, for every $k \in \mathbb{N}$ we can find an extremal graph E_k on at least k vertices such that $\mathrm{RBT} \ge 0.40005$. By compactness, the sequence $(E_k)_{k\in\mathbb{N}}$ has a convergent subsequence and this subsequence converges to some extremal limit e. However, $e(\mathrm{RBT}) \ge 0.40005$, which contradicts Lemma 11. □

5. Proof of Theorem 4

Let G be an extremal graph on n vertices, where n is sufficiently large. Let $Z = \{z_1, z_2, z_3, z_4\}$ be a subset of $V(G)$ such that Z induces an Q, and

$$\mathrm{Q}_{22}(Z) - \tfrac{26}{9}\mathrm{Q}_3(Z) + \tfrac{27}{1000}\mathrm{Q}^+(Z) \tag{11}$$

is maximized over all choices of Z.

Note that in every Q_{22}, four of the 15 vertex subsets of size 4 induce copies of Q, three in every Q_3, and one of the five sets in every Q^+. Since (11) is maximized, we can lower bound it by the average over all $Y \in \mathrm{Q}$ and we obtain

$$\begin{aligned}
&\left(\mathrm{Q}_{22}(Z) - \tfrac{26}{9}\mathrm{Q}_3(Z) + \tfrac{27}{1000}\mathrm{Q}^+(Z)\right)\binom{n-4}{2} \\
\geq\ & \frac{1}{|\mathrm{Q}|}\sum_{Y\in\mathrm{Q}}\left(\left(\mathrm{Q}_{22}(Y) - \tfrac{26}{9}\mathrm{Q}_3(Y)\right)\binom{n-4}{2} + \tfrac{27}{2000}\mathrm{Q}^+(Y)\binom{n-4}{1}(n-5)\right) \\
\geq\ & \frac{\left(4\mathrm{Q}_{22} - 3\cdot\tfrac{26}{9}\mathrm{Q}_3\right)\binom{n}{6} + \tfrac{27}{2000}\mathrm{Q}^+\binom{n}{5}(n-5)}{\mathrm{Q}\binom{n}{4}} \\
=\ & \frac{\tfrac{4}{15}\mathrm{Q}_{22} - \tfrac{26}{45}\mathrm{Q}_3 + \tfrac{27}{5000}\mathrm{Q}^+}{\mathrm{Q}}\binom{n-4}{2}.
\end{aligned}$$

Using (5) and (7), this gives

$$\mathrm{Q}_{22}(Z) - \tfrac{26}{9}\mathrm{Q}_3(Z) + \tfrac{27}{1000}\mathrm{Q}^+(Z) > 0.02760856. \tag{12}$$

For $1 \leq i \leq 4$, we define sets of vertices Z_i which look like z_i to the other vertices of Z. Formally,

$$Z_i := \{v \in V(G) : G[(Z \setminus z_i) \cup v] \cong \mathrm{Q}\} \text{ for } 1 \leq i \leq 4.$$

Note that $Z_i \cap Z_j = \emptyset$ for $i \neq j$. We call an edge v_iv_j *funky*, if v_iv_j is in a different part than z_iz_j, where $v_i \in Z_i$, $v_j \in Z_j$, $1 \leq i < j \leq 4$. In other words, $v_i \neq z_i$, $v_j \neq z_j$ and $G[Z \cup \{v_i, v_j\}] \ncong \mathrm{Q}_{22}$, i.e., every funky edge destroys a potential copy of $\mathrm{Q}_{22}(Z)$. Denote by E_f the set of funky edges. With this notation, for sufficiently large n the inequality (12) implies that

$$\begin{aligned}
&2\sum_{1\leq i<j\leq 4}|Z_i||Z_j| - 2|E_f| - \tfrac{26}{9}\sum_{1\leq i\leq 4}|Z_i|^2 + \tfrac{27n}{1000}\left(n - \sum_{1\leq i\leq 4}|Z_i|\right) \\
&> 0.02760856 \times 2\binom{n-4}{2},
\end{aligned}$$

as we are counting over all pairs of vertices $x, y \notin \{z_1, z_2, z_3, z_4\}$, considering all cases of membership in the Z_i, the funkiness of the edge xy, and the resulting graphs induced on $\{x, y, z_1, z_2, z_3, z_4\}$.

For $X_i \subseteq Z_i$, where $1 \leq i \leq 4$, let $X_0 := V(G) \setminus \bigcup X_i$. Let f be the number of funky edges not incident to vertices in X_0, divided by n^2 for normalization, and denote $x_i = \frac{1}{n}|X_i|$ for $0 \leq i \leq 4$. Choose X_is such that the left hand side of

$$2\sum_{1\leq i<j\leq 4}x_ix_j - 2f - \tfrac{26}{9}\sum_{1\leq i\leq 4}x_i^2 + \tfrac{27}{1000}x_0 > 0.02760856 \tag{13}$$

is maximized.

From this, it is not difficult to check (see Appendix A) that

$$x_0 < 0.0059605; \tag{14}$$

$$0.244287 < \ x_i < 0.255713 \quad \text{for} \quad 1 \le i \le 4; \tag{15}$$

$$0.493403 < \ x_i + x_j < 0.506597 \quad \text{for} \quad 1 \le i < j \le 4; \tag{16}$$

$$f < 0.000084609; \tag{17}$$

$$-\tfrac{25}{27}x_1 + 2x_i - \tfrac{1}{3}x_2 - \tfrac{1}{3}x_3 - \tfrac{1}{3}x_4 < 0.0315 \quad \text{for} \quad 2 \le i \le 4; \tag{18}$$

$$2x_1 - x_2 + x_3 - x_0 > 0.484987; \tag{19}$$

$$x_i + x_0 < 0.2563 \quad \text{for} \quad 1 \le i \le 4. \tag{20}$$

By symmetry, (18) and (19) hold also after permuting the variables. However, we use them explicitly only in this permutation. Furthermore, for any vertex $v \in X_i$ we use $d_f(v)$ to denote the number of funky edges from v to $(X_1 \cup X_2 \cup X_3 \cup X_4) \setminus X_i$ after normalizing by n. We will later also use $d_f(v, X)$ to denote the number of funky edges from v to $X \subseteq (X_1 \cup X_2 \cup X_3 \cup X_4) \setminus X_i$ after normalizing by n. The contribution of $v \in X_1$ to (13) is

$$\frac{1}{n}\left(2(x_2 + x_3 + x_4) - 2d_f(v) - 2 \cdot \tfrac{26}{9}x_1 + o(1)\right).$$

If this quantity was negative, (13) could be increased by moving v to X_0, contradicting our choice of X_i. This and (15) imply that

$$d_f(v) \le x_2 + x_3 + x_4 - \tfrac{26}{9}x_1 + o(1) \le 1 - \tfrac{35}{9}x_1 + o(1) < 0.049995, \tag{21}$$

and symmetric statements hold for $v \in X_2, X_3, X_4$.

Now we use the copy Z of Q to give names to the parts of $E(G)$. Let *green* be the part containing the edge z_1z_2, *red* the part containing z_1z_3, and *blue* the part containing z_1z_4. For brevity, we say that an edge is colored green/red/blue if it belongs to the corresponding part. A re-coloring of an edge is equivalent to moving the edge to a different part of $E(G)$. Note that the non-funky edges of G are colored as in Fig. 3.

Next, we will prove that a vertex $v \in X_i$ cannot be adjacent to almost all vertices of X_i by edges of only one color. For a vertex $v \in V(G)$, we denote by $r_i(v)$, $b_i(v)$ and $g_i(v)$ the numbers of red, blue and green edges from v to X_i, divided by n. Similarly, let $r(v)$, $b(v)$, and $g(v)$ be the numbers of all red/blue/green edges incident to v, divided by n.

Claim 12. *For every $v \in X_i$, we have $x_i - r_i(v), x_i - b_i(v), x_i - g_i(v) > 0.033$, where $i \in \{1, 2, 3, 4\}$.*

Proof. Without loss of generality, let us assume $v \in X_1$ and $x_1 - r_1(v) \le 0.033$. Denote $x_{\max} := \max\{x_2, x_3, x_4\}$. We bound the number of rainbow triangles containing v divided by n^2, i.e., $\left(\frac{1}{2} - o(1)\right) \mathrm{RBT}(v)$, from above. For a rainbow triangle uvw, we distinguish several cases.

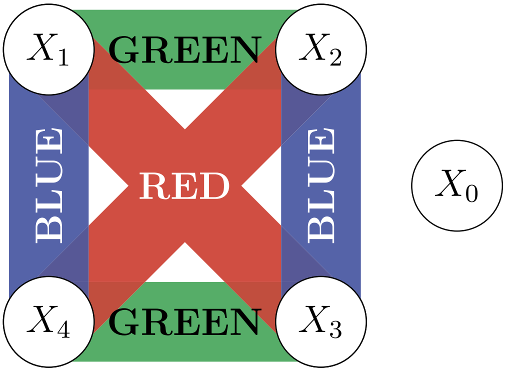

Fig. 3. Coloring of the non-funky edges.

1. If $u, w \in X_1$, then the normalized number of rainbow triangles uvw can be upper bounded by $r_1(v)b_1(v) + r_1(v)g_1(v) + g_1(v)b_1(v)$. This is maximized when $g_1(v) = b_1(v) = \frac{1}{2}(x_1 - r_1(v))$, which gives the upper bound $(r_1(v) + \frac{x_1 - r_1(v)}{4})(x_1 - r_1(v))$ for triangles of this type.
2. If $u \in X_i$ and $w \in X_j$, where $1 < i < j \le 4$, and all of uv, vw, uw are non-funky, then we obtain the upper bound $(x_2x_3 + x_2x_4 + x_3x_4) - d_f(v)(x_2 + x_3 + x_4 - x_{\max}) + \frac{1}{3}d_f(v)^2$ for triangles of this type. The first term counts all possible such triangles, the second term subtracts those which contain a funky edge incident to v (which is minimized if all funky edges from v lead to the X_i of maximum size), and the third term accounts for possible double counting in the second term (which is maximized if the funky edges from v are evenly distributed among the X_i).
3. If uw is a funky edge, then uvw might be rainbow and in this case we get the easy upper bound f for triangles of this type.
4. If $u \in X_0$ then w can be anywhere, which gives the bound x_0 for triangles of this type.
5. We can bound the number of rainbow triangles where both vu and vw are funky by $\frac{1}{3}d_f(v)^2$. The $\frac{1}{3}$ in the term comes from the fact that vu and vw must have different colors for the triangle to be rainbow.
6. If vu is funky and $w \in X_1$, then we get an upper bound of $d_f(v)r_1(v)$ for triangles of this type.
7. If vu is funky and u and w are in the same X_i (for $i \ge 2$), we get an upper bound of $d_f(v)x_{\max}$ for triangles of this type.

Note that it cannot happen that only vu is funky, $v \in X_i$, and $w \in X_j$, where $i, j \in \{2, 3, 4\}$ and $i \ne j$.

Counting all types together, we obtain

$$\frac{1}{2}\mathrm{RBT}(v) \le \left(r_1(v) + \frac{x_1 - r_1(v)}{4}\right)(x_1 - r_1(v)) + x_2x_3 + x_2x_4 + x_3x_4 \\ + f + x_0 + d_f(v)(2x_{\max} + r_1(v) - x_2 - x_3 - x_4 + \tfrac{2}{3}d_f(v)) < 0.1991, \tag{22}$$

which contradicts Corollary 9. The last inequality can be obtained by maximizing (22), which we defer to Appendix B. □

Let us call a vertex $v \in X_i$ *blue* if $x_i - b_i(v) \leq 0.075$, and similarly *red* or *green*, and finally *black* if it has none of the other colors. Note that each vertex has exactly one of the four colors as $x_i > 2 \times 0.075$.

Claim 13. *If $v \in X_1$ is black, then $d_f(v) = 0$.*

Proof. Let vw be a funky edge, and suppose that w is chosen such that $d_f(w)$ is minimized over all funky edges vw. Therefore,

$$d_f(v) \times d_f(w) \leq \sum_{vy \in E_f} d_f(y) \leq 2f,$$

as the sum counts every funky edge in G at most twice. By symmetry, we may assume that $w \in X_2$ and vw is red. As G has maximal rainbow triangle density, recoloring vw to green (making it not funky) can only reduce the number of rainbow triangles. So let us bound the number of rainbow triangles containing vw before and after the recoloring.

$$\begin{aligned} \text{Before:}\quad \mathrm{RBT}(vw) &\leq d_f(v) + d_f(w) + b_1(v) + b_2(w) + x_0; \\ \text{After:}\quad \mathrm{RBT}(vw) &\geq x_3 + x_4 - d_f(v, X_3 \cup X_4) - d_f(w, X_3 \cup X_4) \\ &\geq x_3 + x_4 - d_f(v) - d_f(w). \end{aligned}$$

By the assumption that $\mathrm{RBT}(vw)$ does not increase when the color of vw is changed, we obtain that

$$-b_1(v) \leq b_2(w) - x_3 - x_4 + x_0 + 2d_f(v) + 2d_f(w). \tag{23}$$

By Claim 12, $b_2(w) \leq x_2 - 0.033$, which together with v being black gives

$$\begin{aligned} 0.075 \leq x_1 - b_1(v) &\leq x_1 + x_2 - 0.033 - x_3 - x_4 + x_0 + 2d_f(v) + 2d_f(w) \\ &\leq 2(x_1 + x_2 + x_0) - 0.033 - 1 + 2d_f(v) + 2d_f(w). \end{aligned}$$

Let us maximize the right hand side using (15), (17) and (21). Note that in all formal programs (P) in this paper, the variables appearing in the program are real numbers, and they do not inherit any meaning from outside the program.

$$(P)\begin{cases}\text{maximize} & 2(x_1+x_2+x_0)-0.033-1+2d_f(v)+2d_f(w)\\ \text{subject to} & d_f(v)\times d_f(w)\le 2f\le 2\times 0.000084609,\\ & 0\le d_f(v)\le 1-\frac{35}{9}x_1,\\ & 0\le d_f(w)\le 1-\frac{35}{9}x_2,\\ & 0\le x_0,\\ & 0.244287\le x_1\le 0.255713,\\ & 0.244287\le x_2\le 0.255713.\end{cases}$$

In order to simplify the computation and writeup, we omit the $o(1)$ term that is coming from constraints given by (21). The only change is that the objective functions in the following programs contain $+o(1)$.

To break the symmetry of (P) we assume that $x_1 \le x_2$, making the bound on $d_f(w)$ lower than the bound on $d_f(v)$. This is allowed as all the relations of (P) are symmetric in x_1 and x_2. If x_0, x_1, x_2 are fixed, the maximum of (P) is attained when $d_f(v)$ is maximized, i.e., for $d_f(v) = 1 - \frac{35}{9}x_1$, and then $d_f(w)$ is maximized, i.e., for $d_f(w) = \min\{1 - \frac{35}{9}x_2, 2(0.000084609/d_f(v)\}$.

It follows from (20) that $x_2 + x_0 < 0.2563$, which gives the following relaxation (P_1) of (P) with only one variable:

$$(P_1)\begin{cases}\text{maximize} & 2(x_1+0.2563)-0.033-1+2(1-\frac{35}{9}x_1)\\ & \qquad +4(0.000084609/(1-\frac{35}{9}x_1))\\ \text{subject to} & 0.244287\le x_1\le 0.255713.\end{cases}$$

Simplification of the objective function in (P_1) gives (P_1')

$$(P_1')\begin{cases}\text{maximize} & 1.4796-\frac{52}{9}x_1+0.003045924/(9-35x_1)\\ \text{subject to} & 0.244287\le x_1\le 0.255713.\end{cases}$$

The maximum of P_1' is when $x_1 = 0.244287$ and gives $0.075 > x_1 - b_1(v)$ which contradicts $x_1 - b_1(v) \ge 0.075$. $\square$

Claim 14. *If $v \in X_1 \cup \dots \cup X_4$ is a vertex of color c that is not black, then v is not incident to any funky edges colored c or to funky edges whose non-funky color would be c. For example, a blue vertex $v \in X_1$ can be incident only to funky edges that are not blue and have the other endpoint in X_2 or X_3, in other words, $b_2(v) + b_3(v) + g_4(v) + r_4(v) = 0$.*

Proof. We assume without loss of generality that $v \in X_1$ is blue. Suppose for contradiction that there is a vertex w such that vw is funky and either $w \in X_4$ or if $w \in X_2 \cup X_3$ then vw is blue. Let us only look at the case that $w \in X_2$ and vw blue, the other cases are similar.

By similar arguments as in Claim 13 we count the number of rainbow triangles containing uw and the number after recoloring uw to green. We obtain

$$\begin{aligned}\text{Before:}\quad & \mathrm{RBT}(vw) \leq d_f(v) + d_f(w) + r_1(v) + r_2(w) + x_0;\\ \text{After:}\quad & \mathrm{RBT}(vw) \geq x_3 + x_4 - d_f(v, X_3 \cup X_4) - d_f(w, X_3 \cup X_4)\\ & \qquad\qquad \geq x_3 + x_4 - d_f(v) - d_f(w).\end{aligned}$$

Since switching vw to green may not increase the number of RBT, we get an analogue of (23)

$$-r_1(v) \leq d_f(v) + d_f(w) + r_2(w) + x_0 - (x_3 + x_4 - d_f(v) - d_f(w)). \tag{24}$$

Since v is blue, $r_1(v) \leq 0.075$. With (21) and by adding $x_1 + r_1(v)$ to both sides of (24) we get

$$\begin{aligned} x_1(v) &\leq_{(24)} x_1 + r_1(v) + x_2 - x_3 - x_4 + x_0 + 2d_f(v) + 2d_f(w)\\ &\leq_{(21)} r_1(v) + 4 - \tfrac{61}{9}(x_1 + x_2) - x_3 - x_4 + x_0\\ &= r_1(v) + 4 - \tfrac{52}{9}(x_1 + x_2) - (x_0 + x_1 + x_2 + x_3 + x_4) + 2x_0\\ &\leq r_1(v) + 3 - \tfrac{52}{9}0.493403 + 2 \cdot 0.0059605\\ &< 0.162 + r_1(v) \leq 0.237,\end{aligned}$$

which contradicts (15). □

For every $v \in V(G)$ we define $d_{mono}(v) := max\{r(v), g(v), b(v)\}$.

Claim 15. *The number of vertices v with $d_f(v) > 0$ is less than $0.00937n$. This implies that $d_f(v) < 0.00937$ for all vertices in $V \setminus X_0$.*

Proof. Using (8) and the definition of d_{mono} we get

$$\begin{aligned} 0.33343492 > \tfrac{1}{3}\mathrm{TCT} + \mathrm{MONOT} &= \frac{1}{n}\sum_{v\in V}(r(v)^2 + g(v)^2 + b(v)^2) - o(1)\\ &\geq \frac{1}{n}\sum_{v\in V}(d_{mono}(v)^2 + \tfrac{1}{2}(1 - d_{mono}(v))^2) - o(1) \geq \frac{1}{3} - o(1),\end{aligned}$$

and hence

$$0.333435 > \frac{1}{n}\sum_{v\in V}(d_{mono}(v)^2 + \tfrac{1}{2}(1 - d_{mono}(v))^2). \tag{25}$$

By Claim 13, any v with $d_f(v) > 0$ is not black. Without loss of generality we assume $v \in X_1$ is blue, hence $r_4(v) = g_4(v) = 0$ by Claim 14. Then we have

$$d_{mono}(v) \geq b(v) \geq x_1 - 0.075 + x_4 >_{(16)} 0.4184.$$

So

$$d_{mono}(v)^2 + \tfrac{1}{2}(1 - d_{mono}(v))^2 > 0.344188.$$

By this and (25), we conclude that the number of vertices v with $d_f(v) > 0$ can be at most

$$\frac{0.333435 - \frac{1}{3}}{0.344188 - \frac{1}{3}} n < 0.009367n < 0.00937n. \quad \square$$

Claim 16. *For all $v \in X_1 \cup X_2 \cup X_3 \cup X_4$ we have $d_f(v) = 0$.*

Proof. Suppose that vw is funky, say $v \in X_1$, $w \in X_2$, and vw is red. Then, using (23) and the bounds for $d_f(v)$ from Claim 15,

$$\begin{aligned} x_1 - b_1(v) + x_2 - b_2(w) &\leq x_1 + x_2 - x_3 - x_4 + x_0 + 2d_f(v) + 2d_f(w) \\ &\leq_{(16)} 0.506597 - 0.493403 + 0.0059605 + 4 \times 0.00937 \\ &= 0.0566345, \end{aligned}$$

contradicting Claim 12, which implies that $x_1 - b_1(v) + x_2 - b_2(w) \geq 0.066$. $\square$

Next, we want to show that $X_0 = \emptyset$. For this, suppose that there exists $x \in X_0$. We will add x to one of the X_i such that $d_f(x)$ is minimal. By symmetry, we may assume that x is added to X_1. Note that adding a single vertex to X_1 changes the density bounds we used above by at most $o(1)$.

Claim 17. *For every $x \in X_0$, if x was part of X_1 then $d_f(x) \geq 0.0099$.*

Proof. Let xw be a funky edge, where $w \in X_2$. Since G is extremal, making xw not funky cannot increase the number of rainbow triangles which gives a relation analogous to (23).

$$\begin{aligned} \text{Before: } & \text{RBT}(xw) \leq d_f(x) + b_1(x) + b_2(w) + x_0; \\ \text{After: } & \text{RBT}(xw) \geq x_3 + x_4 - d_f(x). \end{aligned}$$

By the assumption that RBT(xw) does not increase when the color of xw is changed, we obtain that

$$-b_1(x) - b_2(w) \leq -x_3 - x_4 + x_0 + 2d_f(x). \tag{26}$$

We also use the trivial bounds $b_1(x) \leq x_1$ and $b_2(w) \leq x_2 - 0.033$. Then

$$-x_1 - (x_2 - 0.033) \le -b_1(x) - b_2(w) \le_{(26)} -x_3 - x_4 + x_0 + 2d_f(x),$$

$$2d_f(x) \ge x_3 + x_4 + 0.033 - (x_0 + x_1 + x_2) = x_3 + x_4 + 0.033 - (1 - x_3 - x_4)$$
$$= 2x_3 + 2x_4 - 0.967 >_{(16)} 0.019802 > 2 \times 0.0099. \quad \square$$

Using yet a different way of bounding $d_f(x)$ and combining it with Claim 17 we get the following improved bound on $d_f(x)$.

Claim 18. *For every $x \in X_0$, if x was part of X_1, then $d_f(x) > 0.12866$.*

Proof. Suppose for a contradiction that $d_f(x) < 0.12866$. First we derive lower bounds on d_{mono} of vertices in funky edges containing x. Suppose that xw is funky, say $w \in X_2$ and xw is red. By arguments very similar to the proof of Claim 13, we have

$$\text{Before:}\quad \mathrm{RBT}(xw) \le b_1(x) + b_2(w) + g_3(x) + x_0;$$
$$\text{After:}\quad \mathrm{RBT}(xw) \ge x_3 + x_4 - (d_f(x) - r_2(x)).$$

We conclude that

$$b_2(w) \ge x_3 + x_4 - x_0 - b_1(x) - d_f(x) - g_3(x) + r_2(x).$$

Next, we give a lower bound on $d_{mono}(w)$:

$$\begin{aligned} d_{mono}(w) \ge b(w) = b_2(w) + x_3 &\ge 2x_3 + x_4 - x_0 - b_1(x) - d_f(x) - g_3(x) + r_2(x) \\ &>_{(19)} 0.484987 + x_1 - b_1(x) - d_f(x) - g_3(x) + r_2(x) \\ &\ge 0.484987 - d_f(x) - g_3(x) + r_2(x). \end{aligned}$$

Similar bounds hold for all other funky edges incident to x. We give only a conclusion here:

$$d_{mono}(w) \ge \begin{cases} 0.484987 - d_f(x) - g_3(x) + r_2(x) & \text{if } w \in X_2 \text{ and } xw \text{ is red;} \\ 0.484987 - d_f(x) - g_4(x) + b_2(x) & \text{if } w \in X_2 \text{ and } xw \text{ is blue;} \\ 0.484987 - d_f(x) - r_2(x) + g_3(x) & \text{if } w \in X_3 \text{ and } xw \text{ is green;} \\ 0.484987 - d_f(x) - r_4(x) + b_3(x) & \text{if } w \in X_3 \text{ and } xw \text{ is blue;} \\ 0.484987 - d_f(x) - b_3(x) + r_4(x) & \text{if } w \in X_4 \text{ and } xw \text{ is red;} \\ 0.484987 - d_f(x) - b_2(x) + g_4(x) & \text{if } w \in X_4 \text{ and } xw \text{ is green.} \end{cases} \tag{27}$$

Observe that the bound when $w \in X_2$ and xw is red contains the same variables as if $w \in X_3$ and xw is green. The same is true also for $w \in X_2$ with blue xw and $w \in X_4$ with green xw and also for the last pair. In order to fit the following computation on one page, we write it only for the first pair. For the other two pairs, we use analogous operations. It follows from (25), (27) and $d_f(x) = r_2(x) + g_3(x) + b_2(x) + g_4(x) + b_3(x) + r_4(x)$ that

$$
\begin{aligned}
0.333435 >_{(25)} \; & \frac{1}{n}\sum_{v\in V(G)} (d_{mono}(v)^2 + \tfrac{1}{2}(1-d_{mono}(v))^2) \\
& \geq \tfrac{1}{3}(1-d_f(x)) + \tfrac{1}{2}d_f(x) \\
& \quad + r_2(x)[\tfrac{3}{2}(0.484987 - d_f(x) - g_3(x) + r_2(x))^2 \\
& \quad - (0.484987 - d_f(x) - g_3(x) + r_2(x))] \\
& \quad + g_3(x)[\tfrac{3}{2}(0.484987 - d_f(x) + g_3(x) - r_2(x))^2 \\
& \quad - (0.484987 - d_f(x) + g_3(x) - r_2(x))] \\
& \quad + b_2(x)(...) + g_4(x)(...) + b_3(x)(...) + r_4(x)(...) \\
& = \tfrac{1}{3}(1-d_f(x)) + \tfrac{1}{2}d_f(x) \\
& \quad + d_f(x)(\tfrac{3}{2}(0.484987 - d_f(x))^2 - 0.484987 + d_f(x)) \\
& \quad + r_2(x)[3(0.484987 - d_f(x))(r_2(x) - g_3(x)) \\
& \quad + \tfrac{3}{2}(r_2(x) - g_3(x))^2 - (r_2(x) - g_3(x))] \\
& \quad + g_3(x)[3(0.484987 - d_f(x))(g_3(x) - r_2(x)) \\
& \quad + \tfrac{3}{2}(g_3(x) - r_2(x))^2 - (g_3(x) - r_2(x))] + \cdots \\
& = \tfrac{1}{3}(1-d_f(x)) + \tfrac{1}{2}d_f(x) + d_f(x)(\tfrac{3}{2}(0.484987 - d_f(x))^2 - 0.484987 + d_f(x)) \\
& \quad + r_2(x)[3(0.484987 - d_f(x)) - 1)(r_2(x) - g_3(x)) + \tfrac{3}{2}(r_2(x) - g_3(x))^2] \\
& \quad + g_3(x)[3(0.484987 - d_f(x)) - 1)(g_3(x) - r_2(x)) + \tfrac{3}{2}(g_3(x) - r_2(x))^2] + \cdots \\
& = \tfrac{1}{3}(1-d_f(x)) + \tfrac{1}{2}d_f(x) + d_f(x)(\tfrac{3}{2}(0.484987 - d_f(x))^2 - 0.484987 + d_f(x)) \\
& \quad + (3(0.484987 - d_f(x)) - 1)(r_2(x) - g_3(x))^2 \\
& \quad + \tfrac{3}{2}(r_2(x) - g_3(x))^2(r_2(x) + g_3(x)) + \cdots .
\end{aligned}
$$

If $d_f(x) < 0.12866$, then $3(0.484987 - d_f(x)) - 1 > 0$. Hence,

$$
(3(0.484987 - d_f(x)) - 1)(r_2(x) - g_3(x))^2 + \tfrac{3}{2}(r_2(x) - g_3(x))^2(r_2(x) + g_3(x)) \geq 0,
$$

and we can obtain the following lower bound:

$$
\begin{aligned}
0.333435 & \geq \tfrac{1}{3}(1-d_f(x)) + \tfrac{1}{2}d_f(x) + d_f(x)(\tfrac{3}{2}(0.484987 - d_f(x))^2 - 0.484987 + d_f(x)) \\
& = \tfrac{3}{2}d_f(x)^3 + (1 - 3\times 0.484987)d_f(x)^2 \\
& \quad + (\tfrac{1}{6} + \tfrac{3}{2}\times 0.484987^2 - 0.484987)d_f(x) + \tfrac{1}{3},
\end{aligned}
$$

so

$$
0 \geq \tfrac{3}{2}d_f(x)^3 - 0.454961 d_f(x)^2 + 0.03449825 d_f(x) - 0.000102.
$$

All $d_f(x)$ that satisfy the last inequality are in $(-\infty, 0.0031) \cup (0.12866, 0.1716)$. Claim 17 implies that $d_f(x)$ is not in $(-\infty, 0.0031)$, hence $d_f(x) > 0.12866$, which is a contradiction to the assumption $d_f(x) < 0.12866$. □

Claim 19. *The set X_0 is empty.*

Proof. We will show that $\mathrm{RBT}(x) < 0.397$ for any $x \in X_0$, contradicting Corollary 9. For the ease of notation, we will write r_i for $r_i(x)$ etc.

$$\begin{aligned}\tfrac{1}{2}\mathrm{RBT}(x) &\leq \tfrac{1}{2}x_0^2 + x_0(1-x_0) + r_1g_1 + r_1b_1 + g_1b_1 + r_2g_2 + r_2b_2 + g_2b_2 + r_3g_3 + r_3b_3 \\ &\quad + g_3b_3 + r_4g_4 + r_4b_4 + g_4b_4 + r_1(b_2+g_4) + b_2g_4 + g_1(b_3+r_4) + b_3r_4 \\ &\quad + b_1(r_2+g_3) + r_2g_3 + g_2(r_3+b_4) + r_3b_4 \\ &\leq_{(*)} \tfrac{1}{2}x_0^2 + x_0(1-x_0) + 0.1945(1-x_0)^2 <_{(14)} 0.1982,\end{aligned}$$

where $(*)$ comes from a massive computation described in Appendix C. This contradiction proves the claim. □

Claim 20. *For n large enough, we have $|X_i| - |X_j| \leq 1$.*

Proof. By symmetry, for a contradiction we assume $|X_1| - |X_2| \geq 2$. Then we move a vertex from X_1 to X_2 and show that doing so increases the number of rainbow triangles. Recall that $\mathrm{RBT}(v)$ denotes the rooted density of RBT at v. Denote

$$F_{\mathrm{avg}}(m) = \frac{1}{m}\sum_{v\in V(G_m)} \mathrm{RBT}(v)\binom{m-1}{2} = 3\frac{F(m)}{m},$$

where G_m is an extremal graph on m vertices. Let

$$l = \lim_{m\to\infty} \frac{F_{\mathrm{avg}}(m)}{\binom{m-1}{2}}.$$

The limit exists since $F_{\mathrm{avg}}(m)/\binom{m-1}{2} = F(m)/\binom{m}{3}$ is non-increasing and lower bounded by 0.4. Corollary 9 implies that $0.40005 \geq l \geq 0.4$. Let $a_i = |X_i| = nx_i$ for $i \in \{1,2,3,4\}$. We delete v from X_1, where $\mathrm{RBT}(v)$ is minimized over vertices in X_1, and add a duplicate w' of $w \in X_2$, where $\mathrm{RBT}(w)$ is maximized over vertices in X_2. We color ww' arbitrarily.

$$\text{Before: } \mathrm{RBT}(v)\binom{n-1}{2} \leq F_{\mathrm{avg}}(a_1) + a_2a_3 + a_2a_4 + a_3a_4,$$

$$\text{After: } \mathrm{RBT}(w')\binom{n-1}{2} \geq F_{\mathrm{avg}}(a_2) + (a_1-1)a_3 + (a_1-1)a_4 + a_3a_4.$$

Since G is extremal, $\mathrm{RBT}(v) \geq \mathrm{RBT}(w')$. Now we estimate $F_{\mathrm{avg}}(a_1) - F_{\mathrm{avg}}(a_2)$. Since $F_{\mathrm{avg}}(m)/\binom{m-1}{2}$ is non-increasing and its limit is l, for n large enough we have

$$F_{\text{avg}}(a_1) = a_1^2 \cdot l/2 + \varepsilon_1 a_1^2, \qquad F_{\text{avg}}(a_2) = a_2^2 \cdot l/2 + \varepsilon_2 a_2^2$$

and $\varepsilon_1 \leq \varepsilon_2 \leq 0.01$. Then we have $F_{\text{avg}}(a_1) - F_{\text{avg}}(a_2) \leq (l/2 + 0.01)(a_1^2 - a_2^2)$ and obtain

$$\begin{aligned}
0 &\leq (\text{RBT}(v) - \text{RBT}(w'))\binom{n-1}{2} \\
&\leq F_{\text{avg}}(a_1) + a_2a_3 + a_2a_4 + a_3a_4 - F_{\text{avg}}(a_2) - (a_1-1)a_3 - (a_1-1)a_4 - a_3a_4 \\
&\leq (0.5l + 0.01)(a_1^2 - a_2^2) - (a_1 - 1 - a_2)(a_3 + a_4) \\
&< 0.22(a_1 - a_2)(a_1 + a_2) - 0.5(a_1 - a_2)(a_3 + a_4) \\
&\leq (a_1 - a_2)(0.22(a_1 + a_2) - 0.5(a_3 + a_4)) < 0,
\end{aligned}$$

which is a contradiction. □

Claim 20 gives a proof of Theorem 4.

Proof of Theorem 4. Let G be an extremal graph on n vertices, where n is sufficiently large, such that Claim 20 holds. Denote $a = |X_1|$, $b = |X_2|$, $c = |X_3|$ and $d = |X_4|$. By Claim 20, a, b, c, d are as equal as possible. Moreover, by Claims 16 and 19, rainbow triangles are either entirely in one X_i for $1 \leq i \leq 4$, or intersect three of the X_i's. It then follows from the extremality of G that

$$F(n) = F(a) + F(b) + F(c) + F(d) + abc + abd + acd + bcd,$$

which completes the proof of the recurrence. Notice that X_1, X_2, X_3, and X_4 satisfy the claimed blow-up property by Claim 16. □

6. Extremal graphs

Now that we know the limit object, we look at the extremal graphs on n vertices. Using a standard blow-up argument, Theorem 6 implies that any 3-edge-partitioned graph G contains at most $(n^3 - n)/15$ rainbow triangles.

Corollary 21. *Every 3-edge-partitioned graph on n vertices contains at most $(n^3 - n)/15$ rainbow triangles.*

Proof. Suppose there exists a 3-edge-partitioned graph G on k vertices with $r = (k^3 - k)/15 + \ell$ rainbow triangles for some $\ell > 0$. Without loss of generality, G is a 3-edge-partitioning of K_n. Let $G_0 := G$ and G_{i+1}, for $i \in \mathbb{N}$, will be obtained by blowing up every vertex of G by a factor k^i and placing G_i inside every blob. It follows that $v(G_i) = k^{i+1}$ and $F(G_i) = k^{3i} \cdot r + k \cdot F(G_{i-1})$. Recall that $F(G_i)$ denotes the number of rainbow triangles in G_i. Expanding the recurrence, it follows that

$$F(G_i) = \sum_{j=0}^{i} k^{3j} \cdot k^{i-j} \cdot r = \frac{k^{3i}\left(k^3 - k + 15\ell\right)}{15} \cdot \sum_{t=0}^{i} \frac{1}{k^{2t}}.$$

Therefore,

$$\lim_{i\to\infty} \frac{F(G_i)}{\binom{v(G_i)}{3}} = \frac{k^2}{k^2-1} \cdot \frac{6 \cdot \left(k^{3i+3} - k^{3i+1} + 15\ell \cdot k^{3i}\right)}{15 \cdot k^{3(i+1)}} = \frac{2}{5} \cdot \left(1 + \frac{15\ell}{k^3 - k}\right).$$

However, any convergent subsequence of $(G_i)_{i\in\mathbb{N}}$ converges to a homomorphism with the density of rainbow triangles equal to $\frac{2}{5} \cdot \left(1 + \frac{15\ell}{k^3-k}\right) > \frac{2}{5}$, which contradicts Theorem 6. □

The iterated blow-up of Q shows that for n being a power of 4, the bound $(n^3 - n)/15$ is best possible. In this case, we show that the iterated blow-up of Q is actually a unique extremal construction.

Proof of Theorem 5. Denote by R^ℓ the $(\ell-1)$-times iterated blow-up of Q, so R^ℓ has 4^ℓ vertices. Theorem 5 is easily seen to be true for $k = 1$, so suppose for a contradiction that there is a graph G on $n = 4^k$ vertices with $F(G) = F(n) = (n^3 - n)/15$ that is not isomorphic to R^k for a minimal $k \geq 2$.

If G has the structure described in Theorem 4, then G is isomorphic to R^k by the minimality of k, a contradiction. Therefore, $V(G)$ cannot be partitioned into four parts X_1, X_2, X_3, X_4 with $|X_i| = 4^{k-1}$ as described in Theorem 4.

Fix an integer ℓ such that $4^\ell > n_0$, where n_0 is taken from the statement of Theorem 4. Let $\overline{G}$ be the graph obtained by blowing up every vertex of G by a factor of 4^ℓ, and inserting R^ℓ in every part. It follows that $\overline{G}$ has $4^{k+\ell}$ vertices, and

$$F(\overline{G}) = n \cdot F\left(R^\ell\right) + F(G) \cdot 4^{3\ell} = \frac{n \cdot 4^{3\ell} - n \cdot 4^\ell + n^3 \cdot 4^{3\ell} - n \cdot 4^{3\ell}}{15} = \frac{4^{3(k+\ell)} - 4^{k+\ell}}{15}.$$

So $\overline{G}$ must be extremal. However, Theorem 4 implies that $\overline{G}$ can be partitioned into four parts $\overline{X}_1, \overline{X}_2, \overline{X}_3, \overline{X}_4$ with $|\overline{X}_i| = 4^{k+\ell-1}$ as described in Theorem 4. Since any two vertices from $V(\overline{G})$ that arise from blowing up the same vertex of G need to be in the same part, the partition $\overline{X}_1, \overline{X}_2, \overline{X}_3, \overline{X}_4$ provides also a partition of the vertices of G. But this is a partition of G into four parts of the same size as described in Theorem 4, a contradiction. □

Acknowledgment

We are grateful to Hong Liu for fruitful discussions at the beginning of the project.

Appendix A. Giving bounds on the x_i

Here we show how to prove (14)–(20). Suppose we want to derive the upper bound from (15). For this, we treat x_0, x_1, x_2, x_3, x_4 and f as real variables, and solve the following program:

$$(P)\begin{cases}\text{maximize} & x_1\\ \text{subject to} & 2\sum_{1\le i<j\le 4} x_ix_j - 2f - \frac{26}{9}\sum_{1\le i\le 4} x_i^2 + \frac{27}{1000}x_0 > 0.02760856,\\ & x_1+x_2+x_3+x_4+x_0 = 1,\\ & x_i \ge 0 \text{ for } i\in\{0,\dots,4\},\\ & f\ge 0.\end{cases}$$

As a quick check, it can be written to a heuristic online solver APMonitor. We provide the source of the program in file `APM.xi.txt`. However, this method may get stuck in local optima, so it does not provide a proof of global maximization.

A rigorous way is to use the method of Lagrange Multipliers. Since we need to solve several of the programs, we implemented the method in Sage. We provide a commented code in file `solve-xi.py`.

Appendix B. The computation in Claim 12

Since we assumed that $x_1 - r_1(v) \le 0.033$, we have $r_1(v) \ge 0.244287 - 0.033$, and the partial derivative of the right hand side of (22) in direction $r_1(v)$ is $\frac{3}{4}x_1 - \frac{3}{2}r_1(v) + d_f(v)$, which is negative. Thus, to maximize the bound, we need to pick $r_1(v)$ minimal, and thus we may assume that $x_1 - r_1(v) = 0.033$.

Next, the coefficient of $d_f(v)$ in (22) is

$$\begin{aligned}&2x_{\max} + r_1(v) - x_2 - x_3 - x_4 + \tfrac{2}{3}d_f(v) = x_1 + 2x_{\max} - x_2 - x_3 - x_4 - 0.033 + \tfrac{2}{3}d_f(v)\\ &\le_{(21)} x_1 + 2x_{\max} - x_2 - x_3 - x_4 - 0.033 + \tfrac{2}{3}(x_2 + x_3 + x_4 - \tfrac{26}{9}x_1 + o(1))\\ &= -\tfrac{25}{27}x_1 + 2x_{\max} - 0.033 - \tfrac{1}{3}x_2 - \tfrac{1}{3}x_3 - \tfrac{1}{3}x_4 + o(1) <_{(18)} 0,\end{aligned}$$

so we may assume that $d_f(v) = 0$, and the right hand side of (22) becomes

$$((x_1 - 0.033) + \tfrac{0.033}{4})0.033 + x_2x_3 + x_2x_4 + x_3x_4 + f + x_0. \tag{28}$$

Now (28) is maximized when $x_2 = x_3 = x_4$ if we fix all the other variables. Note that this choice will not conflict with any other bounds. So we may assume that $x_2 = x_3 = x_4$.

This gives us

$$\tfrac{1}{2}\mathrm{RBT}(v) \le (x_1 - 0.033 + \tfrac{0.033}{4})0.033 + 3x_2^2 + f + x_0,$$

while from (13):

$$6x_1x_2 - \frac{8}{3}x_2^2 - \frac{26}{9}x_1^2 + 0.027x_0 - 2f > 0.02760856.$$

The resulting program we want to solve is

$$(P)\begin{cases} \text{maximize} & (x_1 - 0.033 + \frac{0.033}{4})0.033 + 3x_2^2 + f + x_0 \\ \text{subject to} & 0.02760856 < 6x_1x_2 - \frac{8}{3}x_2^2 - \frac{26}{9}x_1^2 + 0.027x_0 - 2f, \\ & x_1 + 3x_2 + x_0 = 1, \\ & x_1 \geq 0, \\ & x_2 \geq 0, \\ & x_0 \geq 0, \\ & f \geq 0. \end{cases}$$

We give a solution using Lagrange multipliers. We also implemented a script in Sage performing the computation. The script is in file `solve-claim12.py`.

First observe that if $x_1 = 0$ or $x_2 = 0$, then the program is not feasible. Hence $x_1 > 0$ and $x_2 > 0$. We are left with inequalities $x_0 \geq 0$ and $f \geq 0$, which may be tight. Moreover, we always use $x_1 + 3x_2 + x_0 = 1$ for substitution. To solve this, we divide the analysis in four cases, and use Lagrange multipliers again:

Case 1: If $f = 0$ and $x_0 = 0$, this comes down to solving

$$(P)\begin{cases} \text{maximize} & 0.033x_1 + \frac{1}{3}(1 - x_1)^2 - \frac{3}{4}(0.033)^2 \\ \text{subject to} & 0.02760856 < 2x_1(1 - x_1) - \frac{8}{27}(1 - x_1)^2 - \frac{26}{9}x_1^2. \end{cases}$$

The constraint can be simplified to $0.02760856 < -\frac{2}{27}(4 - 35x_1 + 70x_1^2)$. This quadratic program in one variable has the optimal solution $x_1 \approx 0.24424$, and so $\frac{1}{2}\mathrm{RBT}^v < 0.1985$.

Case 2: If $f = 0$ and $x_0 > 0$, it comes down to solving

$$(P)\begin{cases} \text{maximize} & 0.033x_1 + 3x_2^2 - x_1 - 3x_2 + 1 - \frac{3}{4}(0.033)^2 \\ \text{subject to} & 0.02760856 < 6x_1x_2 - \frac{8}{3}x_2^2 - \frac{26}{9}x_1^2 + 0.027(1 - x_1 - 3x_2), \\ & 0.24 \leq x_1 \leq 0.26, \\ & 0.24 \leq x_2 \leq 0.26. \end{cases}$$

Taking gradients, we get

$$\begin{pmatrix} -0.967 \\ -3 + 6x_2 \end{pmatrix} = \lambda \begin{pmatrix} -\frac{52}{9}x_1 + 6x_2 - 0.027 \\ 6x_1 - \frac{16}{3}x_2 - 0.081 \end{pmatrix},$$

which gives $x_1 \approx 0.24662$, $x_2 \approx 0.24936$, and $\frac{1}{2}\mathrm{RBT}^v < 0.19991$ as the only feasible solution.

Case 3: If $f > 0$ and $x_0 = 0$, it comes down to solving

$$(P)\begin{cases}\text{maximize} & 0.033x_1+\frac{1}{3}(1-x_1)^2+f-\frac{3}{4}(0.033)^2\\ \text{subject to} & 0.02760856<2x_1(1-x_1)-\frac{8}{27}(1-x_1)^2-\frac{26}{9}x_1^2-2f.\end{cases}$$

The constraint can be simplified to $0.02760856 < -\frac{2}{27}(4-35x_1+70x_1^2)-2f$. Taking gradients, we get

$$\begin{pmatrix}0.033-\frac{2}{3}+2x_1\\ 1\end{pmatrix}=\lambda\begin{pmatrix}\frac{70}{27}-\frac{280}{27}x_1\\ -2\end{pmatrix},$$

whose solution $x_1 \approx 0.20803$ together with the constraint implies $f < 0$, a contradiction.
Case 4: If $f > 0$ and $x_0 > 0$, it comes down to solving

$$(P)\begin{cases}\text{maximize} & 0.033x_1+3x_2^2+1-x_1-3x_2+f\\ \text{subject to} & 0.02760856<6x_1x_2-\frac{8}{3}x_2^2-\frac{26}{9}x_1^2+0.027(1-x_1-3x_2)-2f.\end{cases}$$

Taking gradients, we get

$$\begin{pmatrix}0.967\\ -3+6x_2\\ 1\end{pmatrix}=\lambda\begin{pmatrix}-\frac{52}{9}x_1+6x_2-0.027\\ 6x_1-\frac{16}{3}x_2-0.081\\ -2\end{pmatrix}.$$

Similarly to the previous case, we again have $f < 0$, a contradiction.

Appendix C. The computation in Claim 19

The term we want to maximize does not include anything from X_0, so we can assume that $x_0 = 0$. Since $r_1 + g_1 + b_1 = x_1$, we can use bounds involving $x_1, \dots, x_4$. First, we use $x_1 + x_2 + x_3 + x_4 = 1$. Then we use the lower bounds for (15) on all x_i. We also use the four bounds implied by Claim 18 (since there are four options where to put x). Finally, we add the bounds $r_i, g_i, b_i \geq 0$. So we solve the following program:

$$(P)=\begin{cases}\text{maximize} & r_1g_1+r_1b_1+g_1b_1+r_2g_2+r_2b_2+g_2b_2\\ & +r_3g_3+r_3b_3+g_3b_3+r_4g_4+r_4b_4+g_4b_4\\ & +r_1(b_2+g_4)+b_2g_4+g_1(b_3+r_4)+b_3r_4\\ & +b_1(r_2+g_3)+r_2g_3+g_2(r_3+b_4)+r_3b_4\\ \text{subject to} & \sum_{i=1}^4 r_i+g_i+b_i=1,\\ & r_i+g_i+b_i\geq 0.244287 \text{ for } i\in\{1,2,3,4\},\\ & r_2+b_2+g_3+b_3+r_4+g_4\geq 0.12866,\\ & r_1+b_1+r_3+g_3+g_4+b_4\geq 0.12866,\\ & g_1+b_1+r_2+g_2+r_4+b_4\geq 0.12866,\\ & r_1+g_1+g_2+b_2+r_3+b_3\geq 0.12866,\\ & r_i,g_i,b_i\geq 0 \text{ for } i\in\{1,2,3,4\}.\end{cases}$$

The optimal solution to the program has value less than 0.1945 and it is achieved at $r_1 \approx 0.03854, g_1 \approx 0.16720, b_1 \approx 0.03854, r_2 = 0, g_2 \approx 0.24670, b_2 = 0, r_3 \approx 0.19243, g_3 = 0, b_3 \approx 0.06658, r_4 \approx 0.06208, g_4 = 0, b_4 \approx 0.18792$.

For each of the bounds, we consider the two cases that the bound is active (i.e. tight) or inactive, giving us a total of 2^{20} cases. In each of the cases, we have to solve a system of linear equations with up to 12 variables, and check the solution for feasibility. Obviously, this is done by a computer using rational arithmetic. We wrote a program in Sage which performs the computation. We reduce the number of programs to solve by eliminating the cases where some sets of constraints cannot be tight at the same time. For example, it is not possible that $r_1 = g_1 = b_1 = 0$ at the same time. Note that feasible solutions with dimension greater than zero will occur again as lower dimensional solutions in cases with more active bounds, so we only have to analyze discrete solutions. We could use symmetries, and we could analyze the feasibility polytope closer to only check the faces which actually appear (the program Polymake [11] can yield this output), reducing the number of cases to check to a few thousand. But we decided to use this brute-force analysis, as this makes it easier to check the code, and the running time is still very reasonable.

The code performing the computation as well as the outputs can be downloaded at http://www.math.uiuc.edu/~jobal/cikk/rbt.

References

[1] R. Baber, Turán densities of hypercubes, arXiv:1201.3587.

[2] R. Baber, J. Talbot, Hypergraphs do jump, Combin. Probab. Comput. 20 (2) (2011) 161–171.

[3] R. Baber, J. Talbot, A solution to the 2/3 Conjecture, SIAM J. Discrete Math. 28 (2) (2014) 756–766.

[4] J. Balogh, M. Delcourt, B. Lidický, C. Palmer, Rainbow copies of C_4 in edge-colored hypercubes, Discrete Appl. Math. 210 (2016) 35–37.

[5] J. Balogh, P. Hu, B. Lidický, H. Liu, Upper bounds on the size of 4- and 6-cycle-free subgraphs of the hypercube, European J. Combin. 35 (2014) 75–85.

[6] J. Balogh, P. Hu, B. Lidický, F. Pfender, Maximum density of induced 5-cycle is achieved by an iterated blow-up of 5-cycle, European J. Combin. 52 (2016) 47–58 (part A).

[7] J. Balogh, P. Hu, B. Lidický, O. Pikhurko, B. Udvari, J. Volec, Minimum number of monotone subsequences of length 4 in permutations, Combin. Probab. Comput. 24 (4) (2015) 658–679.

[8] J. Cummings, D. Král', F. Pfender, K. Sperfeld, A. Treglown, M. Young, Monochromatic triangles in three-coloured graphs, J. Combin. Theory Ser. B 103 (4) (2013) 489–503.

[9] P. Erdős, A. Hajnal, On Ramsey like theorems. Problems and results, in: Combinatorics, Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972, Math. Inst. Appl., Oxford, 1972, pp. 123–140, Inst. Math. Appl., Southend, 1972.

[10] V. Falgas-Ravry, E.R. Vaughan, Turán h-densities for 3-graphs, Electron. J. Combin. 19 (2012) R40.

[11] E. Gawrilow, M. Joswig, Polymake: a framework for analyzing convex polytopes, in: Polytopes—Combinatorics and Computation, Oberwolfach, 1997, in: DMV Sem., vol. 29, Birkhäuser, Basel, 2000, pp. 43–73.

[12] R. Glebov, D. Král', J. Volec, A problem of Erdős and Sós on 3-graphs, Israel J. Math. 211 (1) (2016) 349–366.

[13] A. Grzesik, On the maximum number of five-cycles in a triangle-free graph, J. Combin. Theory Ser. B 102 (5) (2012) 1061–1066.

[14] H. Hatami, J. Hladký, D. Král', S. Norine, A. Razborov, On the number of pentagons in triangle-free graphs, J. Combin. Theory Ser. A 120 (3) (2013) 722–732.

[15] J. Hladký, D. Král', S. Norine, Counting flags in triangle-free digraphs, in: European Conference on Combinatorics, Graph Theory and Applications, EuroComb 2009, in: Electron. Notes Discrete Math., vol. 34, Elsevier Sci. B.V., Amsterdam, 2009, pp. 621–625.
[16] H. Huang, On the maximum induced density of directed stars and related problems, SIAM J. Discrete Math. 28 (1) (2014) 92–98.
[17] D. Král', L. Mach, J.-S. Sereni, A new lower bound based on Gromov's method of selecting heavily covered points, Discrete Comput. Geom. 48 (2) (2012) 487–498.
[18] J. Nešetřil, V. Rödl (Eds.), Mathematics of Ramsey Theory, Algorithms and Combinatorics, vol. 5, Springer-Verlag, Berlin, 1990.
[19] V. Nikiforov, The number of cliques in graphs of given order and size, Trans. Amer. Math. Soc. 363 (3) (2011) 1599–1618.
[20] O. Pikhurko, E.R. Vaughan, Minimum number of k-cliques in graphs with bounded independence number, Combin. Probab. Comput. 22 (6) (2013) 910–934.
[21] A.A. Razborov, Flag algebras, J. Symbolic Logic 72 (4) (2007) 1239–1282.
[22] A.A. Razborov, On the minimal density of triangles in graphs, Combin. Probab. Comput. 17 (4) (2008) 603–618.
[23] A.A. Razborov, Flag algebras: an interim report, in: The Mathematics of Paul Erdős II, Springer, 2013, pp. 207–232.
[24] A.A. Razborov, On the Caccetta–Häggkvist conjecture with forbidden subgraphs, J. Graph Theory 74 (2) (2013) 236–248.
[25] C. Reiher, The clique density theorem, Ann. of Math. (2) 184 (3) (2016) 683–707.
[26] K. Sperfeld, The inducibility of small oriented graphs, arXiv:1111.4813.