Jan Kratochvíl · Mirka Miller
Dalibor Froncek (Eds.)

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Preface

The 25th International Workshop on Combinatorial Algorithms (IWOCA) was held during October 15–17, 2014, in the picturesque harbor town Duluth, located in the south-west corner of Lake Superior in Minnesota, USA. Autumn is a favorite time of the year for visiting Duluth, owing to the amazing range of colors of tree and shrub folliage on display at this time of the year. The IWOCA 2014 Organizing Committee timed the event perfectly!

IWOCA – the workshop that originated 25 years ago as the (Australasian) AWOCA – has over the years established itself as a truly international conference. The name change (to IWOCA) reflected the expanse of the conference beyond local boundaries, motivated by the growing global interest in the conference. The first IWOCA events were still held in Australia in 2007, and the subsequent years brought it to Japan (2008), the Czech Republic (2009), the UK (2010), Canada (2011), India (2012), France (2013), and to the USA this year. During the last six years the proceedings have been published by Springer in the LNCS series.

IWOCA 2014 received 68 submissions, most of them of very high quality. The Program Committee was faced with hard work and sometimes difficult decisions and we regretted that some good papers had to be rejected because of the limited capacity of the conference schedule. In the end, 32 contributed talks were presented during the conference.

We would like to thank all who have sent their submissions and to congratulate all the authors of the accepted papers. We extend special thanks to the distinguished invited speakers Josep Domingo-Ferrer, Pinar Heggernes, Saketh Saurab, and Xuding Zhu. We also thank all the authors who submitted posters for the poster session (which are, however, not included in these proceedings).

Finally, we thank all the members of the Program Committee, all external reviewers, and all the members of the Organizing Committee for all the hard work they have done. While all committee members worked well as a team, some names must be singled out: Special thanks go to Sergei Bezrukov for tirelessly updating the website and running the technology support during the workshop, and to Xiaofeng Gu for handling technical issues of papers included in both the pre-workshop proceedings and this volume.

March 2015

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3-Coloring Triangle-Free Planar Graphs with a Precolored 9-Cycle

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Abstract. Given a triangle-free planar graph G and a cycle C of length 9 in G, we characterize all situations where a 3-coloring of C does not extend to a proper 3-coloring of G. This extends previous results for the length of C up to 8.

1 Introduction

Let $[n] = \{1, 2, \ldots, n\}$. Graphs in this paper are finite and may have loops or parallel edges. Given a graph G, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. We will also use $|G|$ for the size of $E(G)$. A *proper k-coloring* of a graph G is a function $\varphi : V(G) \to [k]$ such that $\varphi(u) \neq \varphi(v)$ for each edge $uv \in E(G)$. A graph is *k-colorable* if there exists a proper k-coloring of the graph, and the minimum k where a graph is k-colorable is the *chromatic number* of the graph.

Garey and Johnson [15] proved that deciding if a graph is k-colorable is NP-complete even when $k = 3$. Moreover, deciding if a graph is 3-colorable is still NP-complete when restricted to planar graphs [9]. Therefore, even though planar graphs are 4-colorable by the celebrated Four Color Theorem [4,5,19], finding sufficient conditions for a planar graph to be 3-colorable has been an active area of research. A landmark result in this area is Grötzsch's Theorem [17], which is the following:

Theorem 1 ([17]). *Every triangle-free planar graph is 3-colorable.*

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We direct readers to a nice survey by Borodin [7] for more results and conjectures regarding 3-coloring planar graphs.

A graph G is *k-critical* if it is not $(k-1)$-colorable but every proper subgraph of G is $(k-1)$-colorable. Critical graphs are important since they are (in a certain sense) the minimal obstacles in reducing the chromatic number of a graph. Numerous coloring algorithms are based on detecting critical subgraphs. Despite its importance, there is no known characterization of k-critical graphs when $k \geq 4$. On the other hand, there has been some success regarding 4-critical planar graphs. Extending Theorem 1, the Grünbaum–Aksenov Theorem [1,6,18] states that a planar graph with at most three triangles is 3-colorable, and we know that there are infinitely many 4-critical planar graphs with four triangles. Borodin, Dvořák, Kostochka, Lidický, and Yancey [8] were able to characterize all 4-critical planar graphs with four triangles.

Given a graph G and a proper subgraph C of G, we say G is *C-critical for k-coloring* if for every proper subgraph H of G where $C \subseteq H$, there exists a proper k-coloring of C that extends to a proper k-coloring of H, but does not extend to a proper k-coloring of G. Roughly speaking, a C-critical graph for k-coloring is a minimal obstacle when trying to extend a proper k-coloring of C to a proper k-coloring of the entire graph. Note that $(k+1)$-critical graphs are exactly the C-critical graphs for k-coloring with C being the empty graph.

In the proof of Theorem 1, Grötzsch actually proved that any proper coloring of a 4-cycle or a 5-cycle extends to a proper 3-coloring of a triangle-free planar graph. This implies that there are no triangle-free planar graphs that are C-critical for 3-coloring when C is a face of length 4 or 5. This sparked the interest of characterizing triangle-free planar graphs that are C-critical for 3-coloring when C is a face of longer length. Since we deal with 3-coloring triangle-free planar graphs in this paper, from now on, we will write "C-critical" instead of "C-critical for 3-coloring" for the sake of simplicity.

The investigation was first done on planar graphs with girth 5. Walls [22] and Thomassen [20] independently characterized C-critical planar graphs with girth 5 when C is a face of length at most 11. The case when C is a 12-face was initiated in [20], but a complete characterization was given by Dvořák and Kawarabayashi in [11]. Moreover, a recursive approach to identify all C-critical planar graphs with girth 5 when C is a face of any given length is given in [11]. Dvořák and Lidický [10] implemented an algorithm and used a computer to generate all C-critical graphs with girth 5 when C is a face of length at most 16. The graphs generated were then used to reveal some structure of 4-critical graphs on surfaces without short contractible cycles.

The situation for planar graphs with girth 4, which are triangle-free planar graphs, is more complicated since the list of C-critical graphs is not finite when C has size at least 6. We already mentioned that there are no C-critical triangle-free planar graphs when C is a face of length 4 or 5. An alternative proof of the case when C is a 5-face was given by Aksionov [1]. Gimbel and Thomassen [16] not only showed that there exists a C-critical triangle-free planar graph when C is a 6-face, but also characterized all of them. Aksenov, Borodin, and Glebov [2] independently proved the case when C is a 6-face using the discharging method, and also characterized all C-critical triangle-free planar graphs when C is a 7-face in [3]. Dvořák and Lidický [14] used properties of nowhere-zero flows to give simpler proofs of the case when C is either a 6-face or a 7-face, and also characterized C-critical triangle-free planar graphs when C is an 8-face. The case where C is a 7-face was used in [8].

In this paper, we push the project further and characterize all C-critical triangle-free planar graphs when C is a face of length 9. For a plane graph G, let $S(G)$ denote the set of multisets of possible lengths of internal faces of G with length at least 5.

Theorem 2. *Let G be a connected plane triangle-free graph with outer face bounded by a cycle C of length 9. The graph G is C-critical for 3-coloring if and only if G contains no separating cycles of length at most five, the interior of every non-facial 6-cycle contains only faces of length four and one of the following propositions is satisfied (see Fig. 1 for an illustration):*

(a) $S(G) = \{5\}$ *and the 5-face of G intersects C in a path of length at least two.*
(b) $S(G) = \{7\}$ *and the 7-face of G intersects C in a path of length at least three.*
(c) $S(G) = \{5, 6\}$ *and the 5-face, 6-face, of G intersects C in a path of length at least two, and four, respectively.*
(d) $S(G) = \{5, 6\}$ *and G is depicted as $(d1)$ or $(d2)$ in Fig. 1.*
(e) $S(G) = \{5, 5, 5\}$ *and G is depicted as (Bij) in Fig. 1 for all i, j.*

2 Preliminaries

Our proof of Theorem 2 uses the same method as Dvořák and Lidický [14]. The main idea is to use the correspondence between coloring of a plane graph G and flows in the dual of G. In this paper, we give only a brief description of the correspondence and the lemmas useful in our case. A more detailed and general description can be found in [14].

Let $G^\star$ denote the dual of a plane graph G. Let φ be a proper 3-coloring of the vertices of G by colors $\{1, 2, 3\}$. For every edge uv of G, we orient the corresponding edge e in $G^\star$ in the following way. Let e have endpoints f, h in $G^\star$, where f,v,h is in the clockwise order from vertex u in the drawing of G. The edge e will be oriented from f to h if $(\varphi(u), \varphi(v)) \in \{(1, 2), (2, 3), (3, 1)\}$, and from h to f otherwise.

Since φ is a proper coloring, every edge of $G^\star$ has an orientation. Tutte [21] showed that this orientation of $G^\star$ defines a nowhere-zero $\mathbb{Z}_3$-flow, which means that the in-degree and the out-degree of every vertex in $G^\star$ differ by a multiple of three. Conversely, every nowhere-zero $\mathbb{Z}_3$-flow in $G^\star$ defines a proper 3-coloring of G up to the rotation of colors.

Let h be the vertex in $G^\star$ corresponding to the outer face of G. Edges oriented away from h are called *source edges* and the edges oriented towards h are called *sink edges*. The orientations of edges incident to h depend only on the coloring of C, where C is the cycle bounding the outer face of G.

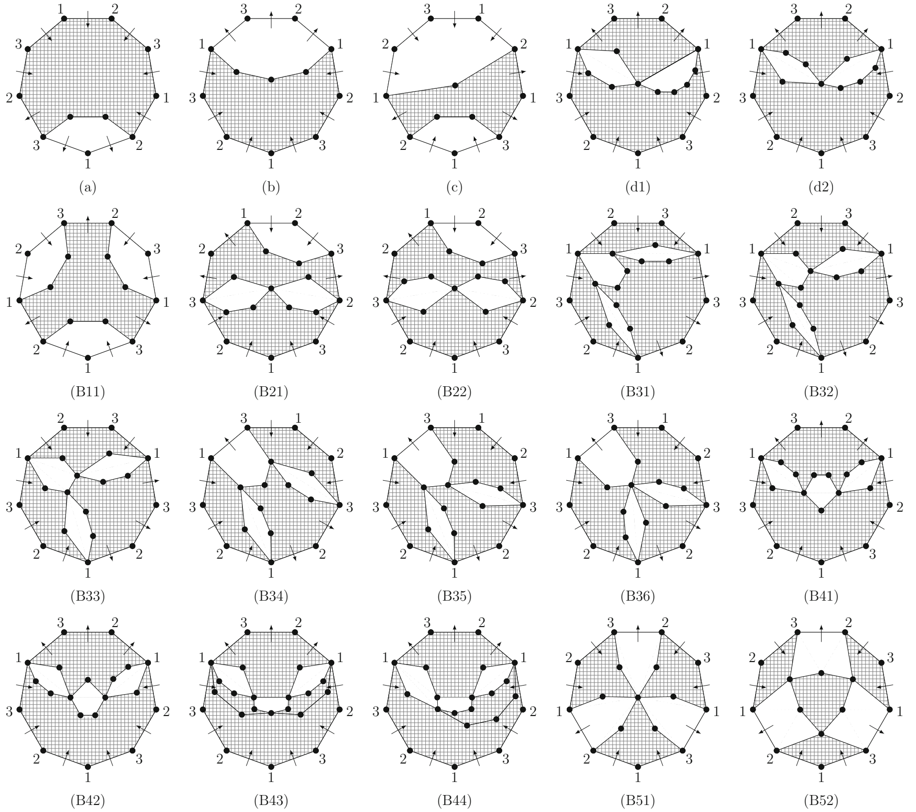

Fig. 1. All C-critical triangle-free plane graphs where C is an outer 9-face. Note that each figure actually represents infinitely many graphs, including ones that can be obtained by identifying some of the depicted vertices. The arrows correspond to source edges and sink edges that are defined in Preliminaries.

For a vertex f of $G^\star$, let $\delta(f)$ denote the difference of the out-degree and in-degree of f. Possible values of $\delta(f)$ depend on the size of the face corresponding to f, denoted by $|f|$. Clearly $|\delta(f)| \leq |f|$ and $\delta(f)$ has the same parity as $|f|$. Hence if $|f| = 4$, then $\delta(f) = 0$. Similarly, if $|f| \in \{5, 7\}$, then $\delta(f) \in \{-3, 3\}$ and if $|f| = 6$ then $\delta(f) \in \{-6, 0, 6\}$.

Next we convert the problem of extending a proper 3-coloring of C to the existence of a $\mathbb{Z}$-flow in an auxiliary graph obtained from $G^\star$. We call a function q assigning an integer to every internal face f of G a *layout* if $q(f) \leq |f|$, $q(f)$ is divisible by 3, and $q(f)$ has the same parity as $|f|$. Notice that $q(f)$ satisfies the same conditions as $\delta(f)$. Therefore it is sufficient to specify the q-values for faces of size at least 5, since $q(f) = 0$ if f is a 4-face.

Let ψ be a proper 3-coloring C. The coloring ψ gives an orientation of the edges corresponding to the edges of C in $G^\star$. Denote by n^s the number of source edges and by n^t the number of sink edges. A layout q is ψ-*balanced* if $n^s + m = n^t$, where m is the sum of the q-values over all internal faces of G. A graph $G^{q,\psi}$ is obtained from $G^\star$ by removing the vertex h corresponding to the outer face of G and by adding two new vertices s and t. For every edge hf in $G^\star$ we add one edge sf if hf is a source edge and we add one edge tf if it is a sink edge. Moreover, for every internal face f with $q(f) > 0$, we add $q(f)$ parallel sf edges and for every internal face f with $q(f) < 0$, we add $-q(f)$ parallel tf edges. Note that q is ψ-balanced if and only if s and t have the same degree.

For a ψ-balanced layout q of G, let $c(q,\psi)$ denote the degree of the source vertex s (and also the sink vertex t) of $G^{q,\psi}$. For an edge cut K in $G^{q,\psi}$ separating s from t, the component of $G^{q,\psi} \setminus K$ containing s, or t, is called a *source component*, or a *sink component*, respectively.

For a set of faces F, let $\ell(F)$ denote the smallest length of a cycle in a critical graph that may contain all faces of F. Denote a face of size i by f_i. It is known [13] that $\ell(\{f_i\}) = i$ and $\ell(\{f_5, f_6\}) = 9$.

The next lemma describes interiors of cycles in critical graphs.

Lemma 1 ([12]). *Let G be a plane graph with outer face K. Let C be a cycle in G that does not bound a face, and let H be the subgraph of G drawn in the closed disk bounded by C. If G is K-critical for k-coloring, then H is C-critical for k-coloring.*

Lemma 2 is the key lemma that gives the correspondence between 3-colorings of C and flows. It implies that if a 3-coloring of C extends to the entire graph, then there is a $\mathbb{Z}$-flow from s to t of value $c(q,\psi)$.

Lemma 2 ([14]). *Let G be a connected plane triangle-free graph with the outer face C bounded by a cycle and let ψ be a 3-coloring of C. The coloring ψ extends to a 3-coloring of G if and only if there exists a ψ-balanced layout q such that the terminals of $G^{q,\psi}$ are not separated by an edge cut smaller than $c(q,\psi)$.*

The cuts showing that a 3-coloring of C does not extend are described by the following lemma.

Lemma 3 ([14]). *Let G be a connected plane triangle-free graph with the outer face C bounded by a cycle and let ψ be a 3-coloring of C that does not extend to a 3-coloring of G. If q is a ψ-balanced layout in G, then there exists a subgraph $K_0 \subseteq G$ such that either*

(i) K_0 is a path with both ends in C and no internal vertex in C, and if P is a path in C joining the end vertices of K_0, n_s is the number of source edges of P, n_t is the number of the sink edges of P and m is the sum of the values of q over all faces of G drawn in the open disk bounded by the cycle $P + K_0$, then $|n_s + m - n_t| > |K_0|$. In particular, $|P| + |m| > |K_0|$.
Or,

(ii) K_0 is a cycle with at most one vertex in C, and if m is the sum of the values of q over all faces of G drawn in the open disk bounded by K_0, then $|m| > |K_0|$.

3 Proof of Theorem 2

Let $\mathcal{S}_k$ be the set of possible multisets of sizes of faces of length at least five in a graph of girth at least 4 where the length of the precolored face is k. The result of Dvořák, Král', and Thomas [13] implies among others that $\mathcal{S}_6 = \{\emptyset\}$, $\mathcal{S}_7 = \{\{5\}\}$, $\mathcal{S}_8 = \{\emptyset, \{6\}, \{5,5\}\}$, and $\mathcal{S}_9 = \{\{7\}, \{5\}, \{6,5\}, \{5,5,5\}\}$.

From now on, G is always a C-critical triangle-free plane graph and C is always the outer face of length 9. By the previous paragraph, we have four cases to consider when C has length 9. The case of one 7-face was already resolved by Dvořák and Lidický [14], and it is described in Theorem 2(b). We resolve the remaining three cases in Lemmas 4, 5, and 6. The proof of Lemma 6 is omitted due to the page limit. In order to simplify the cases, we first solve the case when C has a chord.

If G is C-critical and C has a chord, then Lemma 1 implies that G can be obtained by identifying two edges of the outer faces of two different smaller critical graphs. It is not difficult to show that the converse is also true.

Therefore, we can enumerate C-critical graphs G where C has a chord and has length 9 by identifying edges from two smaller critical graphs with outer faces of length either 4 and 7 or 5 and 6. The resulting graphs are depicted in Fig. 1 (a) and (b), where some of the vertices must be identified.

In the following we assume that C has no chords. In the rest of the paper, ψ will always be a 3-coloring of C. Also, for a subset Z of the edges of C, we will use n^s_Z and n^t_Z to denote the number of source edges and sink edges of Z, respectively.

Lemma 4. *If G contains one 5-face f_5 and one 6-face f_6, and all other faces are 4-faces, then G is described by Theorem 2(c),(d) and depicted in Fig. 1(c),(d1), and (d2).*

Proof. Let G be a C-critical graph containing one 5-face f_5 and one 6-face f_6.

Let $e \in E(G) \setminus E(C)$. We want to find a 3-coloring ψ of C that does not extend to a proper 3-coloring of G but extends to a proper 3-coloring of $G - e$. Note that $G - e$ has either one 5-face and one 8-face, or one 6-face and one 7-face, or one 9-face, or two 6-faces and and one 5-face. We know that the smallest k such that $\mathcal{S}_k$ contains any of $\{5,8\}$, $\{6,7\}$, $\{9\}$, or $\{5,6,6\}$ is at least 11. Hence every precoloring of C extends to $G - e$. In particular, ψ extends to $G - e$. Therefore, we only need to characterize ψ that does not extend to G.

Let ψ be a proper 3-coloring of C that does not extend to a proper 3-coloring of G. By symmetry, we assume that C has more source edges than sink edges. Hence C has either 9 or 6 source edges. Let q be a ψ-balanced layout of G. By Lemma 2, there exists an edge-cut K in $G^{q,\psi}$ separating s from t such that $|K|$ is smaller than $c(q,\psi)$. Let $K_0 \subset G$ be obtained by Lemma 3 and let $k_0 = |K_0|$.

First suppose that K_0 is a cycle. Let m denote the sum of the q-values of the faces in the interior of K_0. By Lemma 3, $|m| > k_0$. If both f_5, f_6 are in the interior of K_0, then $|m| \leq 9$, contradicting the fact that $|m| > k_0$ since $k_0 \geq \ell(\{f_5, f_6\}) = 9$. If f_5 is in the interior of K_0, but f_6 is not, then $|m| = 3$, while $\ell(\{f_5\}) = 5$, a contradiction again. Similarly, we obtain a contradiction when f_6 is in the interior of K_0 but f_5 is not, since $\ell(\{f_6\}) = 6$ and $|m| \leq 6$. Therefore K_0 is always a path joining two distinct vertices of C.

The graph G bounded by C is divided by K_0 into two closed disks X and Y intersecting at K_0, where faces in X correspond to the vertices in the component containing s in $G^{q,\psi} - K$. For $Z \in \{X, Y\}$, denote by P_Z the subpath of C such that Z is bounded by $P_Z + K_0$. Recall that n^s_Z and n^t_Z denote the number of source edges and sink edges in P_Z, respectively. The described structure is shown in Fig. 2.

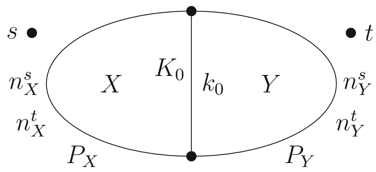

Fig. 2. Structure of a cut in G.

Claim 1. *There are 6 source edges in C.*

Proof. Suppose for a contradiction that C contains 9 source edges. Hence there is just one ψ-balanced layout q with $q(f_5) = -3$, $q(f_6) = -6$, and $c(q, \psi) = 9$. Note that $n^s_X + n^s_Y = 9$ and $n^t_X + n^t_Y = 0$. If both f_5, f_6 belong to X then $|K| = k_0 + n^s_Y + 9 < 9$, a contradiction. If both f_5, f_6 belong to Y then $|K| = k_0 + n^s_Y < 9$, while the length of the boundary cycle of Y is $k_0 + n^s_Y \geq \ell(\{f_5, f_6\}) = 9$, which is a contradiction again. Now suppose that exactly one of f_5, f_6 belongs to X and let f_X denote such a face and f_Y the other one. Then $|K| = k_0 + n^s_Y + |q(f_X)| < 9$ and $k_0 + n^s_Y \geq |f_Y|$. If $f_X = f_5$ then $k_0 + n^s_Y + 3 < 9$ and $k_0 + n^s_Y \geq 6$, which is a contradiction. If $f_X = f_6$ then $k_0 + n^s_Y + 6 < 9$ and $k_0 + n^s_Y \geq 5$, a contradiction. □

Claim 2. *If q is a ψ-balanced layout with $q(f_5) = -3$ and $q(f_6) = 0$, then f_5 belongs to Y and f_6 belongs to X.*

Proof. Assume that $q(f_5) = -3$ and $q(f_6) = 0$. Hence the six source edges are the only edges incident to s, thus $c(q, \psi) = 6$. Note that $n^s_X + n^s_Y = 6$ and $n^t_X + n^t_Y = 3$. First suppose that both f_5, f_6 belong to X. Then $n^s_X + n^t_X + k_0 \geq \ell(\{f_5, f_6\}) = 9$, and the size of the cut K is $3 + k_0 + n^t_X + n^s_Y < c(q, \psi) = 6$. By subtracting the two previous inequalities we get $n^s_X - n^s_Y > 6$, contradicting the fact that $n^s_X + n^s_Y = 6$. Now suppose that both f_5, f_6 belong to Y. Then $n^s_Y + n^t_Y + k_0 \geq \ell(\{f_5, f_6\}) = 9$ and $|K| = k_0 + n^t_X + n^s_Y < 6$. By subtracting them we get $n^t_Y - n^t_X > 3$, a contradiction with $n^t_Y + n^t_X = 3$. Finally we suppose that f_5 belongs to X and f_6 belongs to Y. Then $n^s_Y + n^t_Y + k_0 \geq \ell(\{f_6\}) = 6$ and $|K| = 3 + k_0 + n^t_X + n^s_Y < 6$. But then $n^t_Y - n^t_X > 3$, a contradiction again. Therefore f_5 is in Y and f_6 is in X. □

Claim 3. *If q is a ψ-balanced layout with $q(f_5) = 3$ and $q(f_6) = -6$, then f_5 belongs to X and f_6 belongs to Y.*

Proof. Assume that $q(f_5) = 3$ and $q(f_6) = -6$. Since there are six source edges on C and three edges from s to f_5 in $G^{q,\psi}$, $c(q,\psi) = 9$. Note that $n^s_X + n^s_Y = 6$ and $n^t_X + n^t_Y = 3$. First suppose that both f_5 and f_6 belong to X. Then $k_0 + n^s_X + n^t_X \geq \ell(\{f_5, f_6\}) = 9$ and $|K| = 6 + k_0 + n^s_Y + n^t_X < c(q,\psi) = 9$. But then we obtain $n^s_X - n^s_Y > 6$, contradicting the fact that $n^s_X + n^s_Y = 6$. Now suppose that both f_5 and f_6 belong to Y. Then $k_0 + n^s_Y + n^t_Y \geq \ell(\{f_5, f_6\}) = 9$ and the size of K is $3 + k_0 + n^s_Y + n^t_X < 9$. But then we get $n^t_Y - n^t_X > 3$, contradicting $n^t_X + n^t_Y = 3$. Finally we suppose that f_6 belongs to X and f_5 belongs to Y. Then $|K| = 9 + k_0 + n^s_Y + n^t_X < 9$, a contradiction. $\square$

Since C has 6 source edges, we have two different ψ-balanced layouts. Let q_1 and q_2 be the layouts where $q_1(f_5) = -3$, $q_1(f_6) = 0$, and $q_2(f_5) = 3$, $q_2(f_6) = -6$, respectively. Let K and L be the subgraphs of G obtained by Lemma 3 applied to q_1 and q_2, respectively, and let $k = |K|$ and $l = |L|$. Note that we already showed that each of K and L is a path joining pairs of distinct vertices of C. Denote these vertices by v_1, v_2 for K and by w_1, w_2 for L. The prescribed structure is depicted in Figs. 3 and 4.

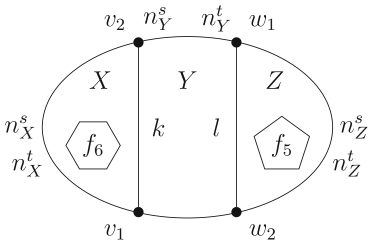

Fig. 3. A structure for two non-crossing cuts.

If we can choose the labels of the endpoints of K and L so that the clockwise order along C is v_1, v_2, w_1, w_2, then we call K and L *non-crossing*, and we call K and L *crossing* otherwise. Notice that K and L are always non-crossing if they have a vertex of C in common.

We treat the cases of K and L being crossing and non-crossing separately.

Claim 4. *If K and L are non-crossing, then G is depicted in Fig. 1(c).*

Proof. Assume that K and L are non-crossing. See Fig. 3. Note that K, L are not necessarily disjoint. The cuts K and L partition G into three parts. Denote by X the region of G containing f_6, by Z the region of G containing f_5, and by Y the rest of G. For an edge cut K' of $G^{q_1,\psi}$ corresponding to K, f_6 belongs to the source subdisk of G while f_5 belongs to the sink subdisk of G by Claim 2.

Analogously, for an edge cut L' of $G^{q_2,\psi}$ corresponding to L, f_5 belongs to the source subdisk of G while f_6 belongs to the sink subdisk of G by Claim 3. For the edge cut K', $|K'| = k + n_X^t + n_Y^s + n_Z^s < c(q_1, \psi) = 6$. For the edge cut L', $|L'| = l + n_X^s + n_Y^s + n_Z^t < c(q_2, \psi) = 9$. By the assumptions that C has no chord, $k \geq 2$ and $l \geq 2$. Since X contains f_6, $k + n_X^s + n_X^t \geq \ell(\{f_6\}) = 6$ and even, and since Z contains f_5, $l + n_Z^s + n_Z^t \geq \ell(\{f_5\}) = 5$ and odd. Clearly $n_X^s + n_Y^s + n_Z^s = 6$ and $n_X^t + n_Y^t + n_Z^t = 3$. Integer solutions to these constraints are in the following table:

n_X^s	n_X^t	n_Y^s	n_Y^t	n_Z^s	n_Z^t	k	l
4	0	0	2	2	1	2	2
4	0	0	3	2	0	2	3

From these solutions we obtain the graphs depicted in Fig. 1(c). □

Claim 5. *If K and L are crossing, then G is depicted in Fig. 1(d1) or (d2).*

Proof. Assume that K and L cross, hence G is divided by K and L into four regions. Let X be the region of G containing f_6, Z be the region containing f_5, and let W,Y be the two remaining regions. Since K and L cross, they have a common internal vertex v. Note that $K \cap L$ is a path and v can be any vertex on the path. Denote by k_1 the length of the subpath of K between X and Y up to v, and denote by k_2 the length of the rest of K. Denote by l_1 the length of the subpath of L between Y and Z up to v, and denote by l_2 the length of the rest of L. The prescribed structure is depicted in Fig. 4.

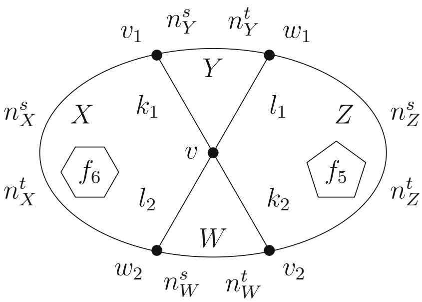

Fig. 4. A structure for two crossed cuts.

Note that $\min\{k_1, k_2, l_1, l_2\} \geq 1$ since v is an internal vertex. For an edge cut K' of $G^{q_1,\psi}$ corresponding to K, f_6 belongs to the source component while f_5 belongs to the sink component by Claim 2. Analogously, for an edge cut L' of $G^{q_2,\psi}$ corresponding to L, f_5 belongs to the source component while f_6 belongs to the sink component by Claim 3.

We obtain the following set of constraints that must be satisfied in this subcase.

$$|K'| = k_1 + k_2 + n_X^t + n_Y^s + n_Z^s + n_W^t < c(q_1, \psi) = 6 \tag{1}$$

$$|L'| = l_1 + l_2 + n_X^s + n_Y^s + n_Z^t + n_W^t < c(q_2, \psi) = 9 \tag{2}$$

$$k_1 + l_2 + n_X^s + n_X^t \geq \ell(\{f_6\}) = 6 \text{ and even} \tag{3}$$

$$l_1 + k_2 + n_Z^s + n_Z^t \geq \ell(\{f_5\}) = 5 \text{ and odd} \tag{4}$$

$$l_2 + k_2 + n_X^s + n_X^t + n_Y^s + n_Y^t + n_Z^s + n_Z^t \geq \ell(\{f_5, f_6\}) = 9 \text{ and odd} \tag{5}$$

$$\min\{k_1, l_1\} + n_Y^s + n_Y^t > \max\{k_1, l_1\} \tag{6}$$

$$\min\{k_2, l_2\} + n_W^s + n_W^t > \max\{k_2, l_2\} \tag{7}$$

$$n_X^s + n_Y^s + n_Z^s = 6 \tag{8}$$

$$n_X^t + n_Y^t + n_Z^t = 3 \tag{9}$$

Inequalities (1) and (2) come from the size of the cut being smaller than $c(q_1, \psi)$ and $c(q_2, \psi)$, respectively. Inequalities (3)–(5) come from the fact that interior of cycles are also critical graphs. Finally, if any of the inequalities (6)–(7) are violated then the cuts K and L can be taken as non-crossing.

We solve the system of constraints by computer programs. From these solutions we get graphs depicted in Fig. 1(d1) and (d2). □

This finishes the proof of Lemma 4. □

Lemma 5. *If G contains one 5-face f_5 and all other faces are 4-faces, then G is described by Theorem 2(a) and depicted in Fig. 1(a).*

Proof. Let G be a C-critical graph containing one 5-face f_5. Let $e \in E(G) \setminus E(C)$. We want to find a 3-coloring ψ of C that does not extend to a proper 3-coloring of G but extends to a proper 3-coloring of $G - e$. Note that either $G - e$ has a 5-face and a 6-face or $G - e$ has a 7-face. This gives us two cases to consider.

Case 1: $G - e$ contains a 5-face and a 6-face.
Let ψ be a 3-coloring of C containing 9 source edges (i.e. the colors around C are $1, 2, 3, 1, 2, 3, 1, 2, 3$). Then ψ extends to a 3-coloring of $G - e$ by Claim 1. However, ψ does not extend to a 3-coloring of G since it is not possible to create a ψ-balanced layout for G.

Case 2: $G - e$ contains a 7-face f_7.
By Theorem 9 from [14], if ψ is a 3-coloring of C containing 9 source edges, then ψ does not extend to a proper 3-coloring of $G - e$ and if ψ is a 3-coloring of C containing 6 source edges and 3 sink edges, then ψ always extends to a proper 3-coloring of $G - e$. Since ψ must extend to $G - e$, we know that ψ contains 6 source edges and 3 sink edges. Now we need to construct such a proper 3-coloring ψ that does not extend to a proper 3-coloring of G.
Let q be a ψ-balanced layout of G. The only possibility is $q(f_5) = -3$ and $c(q, \psi) = 6$. By Lemma 2, there exists an edge-cut K in $G^{q,\psi}$ separating

s from t such that $|K|$ is smaller than 6. By a proof of Lemma 3 (for details see [14]), there is a subgraph K_0 of G containing edges of G, which are crossed by edges of K that are not adjacent to any of the terminals in $G^{q,\psi}$. Denote $|K_0|$ by k_0. First suppose that K_0 is a cycle. Let m denote the sum of the q-values of the faces in the interior of K_0. By Lemma 3 $|m| > k_0$. If f_5 is in the interior of K_0, then $|m| = 3$, while $\ell(\{f_5\}) = 5$, a contradiction. Therefore K_0 is a path joining two distinct vertices of C.

From a ψ-balanced layout q we obtain that $n^s_X + n^s_Y = 6$ and $n^t_X + n^t_Y = 3$. This structure is the same as in the proof of Lemma 4 (see Fig. 2). The following two possibilities can occur:

Let f_5 belong to X. For the edge cut K, $|K| = k_0 + n^s_Y + n^t_X + 3 < c(q,\psi) = 6$. Hence $k_0 = 2$, $n^s_Y = 0$, $n^t_X = 0$, $n^s_X = 6$, and $n^t_Y = 3$. The cycle bounding X has length 8. However, it contains only one face of odd size, which is a contradiction.

Let f_5 belong to Y. For the edge-cut K, $|K| = k_0 + n^s_Y + n^t_X < c(q,\psi) = 6$. For X we have $k_0 + n^s_X + n^t_X \geq \ell(\{f_4\}) = 4$ and even. Since Y contains f_5, $k_0 + n^s_X + n^t_X \geq \ell(\{f_5\}) = 5$ and odd. We solve the system of these constraints by computer programs.

From these solutions we obtain that either Y is a 5-face f_5 sharing at least two sink edges with C (the first three solutions) or Y is bounded by a 7-cycle sharing at least three sink edges with C (the last three solutions). The situation is depicted in Fig. 1(a)(b). □

Lemma 6. *If G contains three 5-faces and all other faces are 4-faces, then G is described by Theorem 2(e) and depicted in Fig. 1(Bij) for all i and j.*

The proof of Lemma 6 is omitted due to the page limit. The proof goes along similar lines as the proof of Lemma 4.

References

1. Aksenov, V.A.: The extension of a 3-coloring on planar graphs. Diskret. Analiz (Vyp. 26 Grafy i Testy), 3–19, 84 (1974)
2. Aksenov, V.A., Borodin, O.V., Glebov, A.N.: Continuation of a 3-coloring from a 6-face to a plane graph without 3-cycles. Diskretn. Anal. Issled. Oper. Ser. 1 **10**(3), 3–11 (2003)
3. Aksenov, V.A., Borodin, O.V., Glebov, A.N.: Continuation of a 3-coloring from a 7-face onto a plane graph without 3-cycles. Sib. Èlektron. Mat. Izv. **1**, 117–128 (2004)
4. Appel, K., Haken, W.: Every planar map is four colorable. I. Discharging. Illinois J. Math. **21**(3), 429–490 (1977)
5. Appel, K., Haken, W., Koch, J.: Every planar map is four colorable. II. Reducibility. Illinois J. Math. **21**(3), 491–567 (1977)
6. Borodin, O.V.: A new proof of Grünbaum's 3 color theorem. Discrete Math. **169**(1–3), 177–183 (1997). http://dx.doi.org/10.1016/0012-365X(95)00984-5

7. Borodin, O.V.: Colorings of plane graphs: A survey. Discrete Math. **33**(4), 517–539 (2013). http://dx.doi.org/10.1016/j.disc.2012.11.011
8. Borodin, O.V., Dvořák, Z., Kostochka, A.V., Lidický, B., Yancey, M.: Planar 4-critical graphs with four triangles. Eur. J. Combin. **41**, 138–151 (2014). http://dx.doi.org/10.1016/j.ejc.2014.03.009
9. Dailey, D.P.: Uniqueness of colorability and colorability of planar 4-regular graphs are NP-complete. Discrete Math. **30**(3), 289–293 (1980). http://dx.doi.org/10.1016/0012-365X(80)90236-8
10. Dvořák, Z., Lidický, B.: 4-critical graphs on surfaces without contractible ($\leq$ 4)-cycles. SIAM J. Discrete Math. **28**(1), 521–552 (2014). http://dx.doi.org/10.1137/130920952
11. Dvořák, Z., Kawarabayashi, K.i.: Choosability of planar graphs of girth 5. ArXiv e-prints, September 2011
12. Dvořák, Z., Král, D., Thomas, R.: Three-coloring triangle-free graphs on surfaces I. Extending a coloring to a disk with one triangle (2013) (Submitted)
13. Dvořák, Z., Král, D., Thomas, R.: Three-coloring triangle-free graphs on surfaces IV. 4-faces in critical graphs (2014) (Manuscript)
14. Dvořák, Z., Lidický, B.: 3-coloring triangle-free planar graphs with a precolored 8-cycle (2014). http://dx.doi.org/10.1002/jgt.21842 (Accepted to J. Graph Theory)
15. Garey, M.R., Johnson, D.S.: Computers and Intractability: A Guide to the Theory of NP-Completeness. W. H. Freeman and Co., San Francisco (1979)
16. Gimbel, J., Thomassen, C.: Coloring graphs with fixed genus and girth. Trans. Amer. Math. Soc. **349**(11), 4555–4564 (1997). http://dx.doi.org/10.1090/S0002-9947-97-01926-0
17. Grötzsch, H.: Ein Dreifarbenzatz für Dreikreisfreie Netze auf der Kugel. Math. Natur. Reihe **8**, 109–120 (1959)
18. Grünbaum, B.: Grötzsch's theorem on 3-colorings. Michigan Math. J. **10**, 303–310 (1963)
19. Robertson, N., Sanders, D., Seymour, P., Thomas, R.: The four-colour theorem. J. Combin. Theory Ser. B **70**(1), 2–44 (1997). http://dx.doi.org/10.1006/jctb.1997.1750
20. Thomassen, C.: The chromatic number of a graph of girth 5 on a fixed surface. J. Combin. Theory Ser. B **87**(1), 38–71 (2003). http://dx.doi.org/10.1016/S0095-8956(02)00027-8 (dedicated to Crispin St. J. A. Nash-Williams)
21. Tutte, W.T.: A contribution to the theory of chromatic polynomials. Canadian J. Math. **6**, 80–91 (1954)
22. Walls, B.H.: Coloring girth restricted graphs on surfaces. ProQuest LLC, Ann Arbor (1999). http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqdiss&rft_dat=xri:pqdiss:9953838 (thesis (Ph.D.)–Georgia Institute of Technology)