Discharging and List coloring

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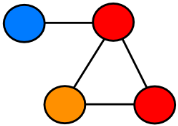
Outline

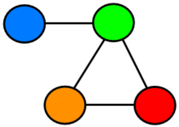
1. List coloring
   - From Coloring to List Coloring
   - Coloring vs. List Coloring
2. Discharging
   - What is discharging?
   - Example

Graph Coloring



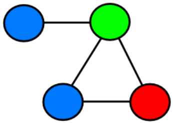
Definition

The coloring is assignment a color to every vertex.



Definition

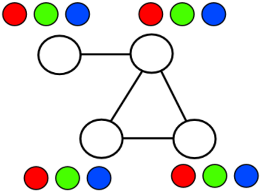
The proper coloring is a coloring where adjacent vertices have different colors.



Definition

The chromatic number of graph is minimal number of colors needed by a proper coloring. Denoted by $\chi(G)$.

Generalizing The Graph Coloring



- Coloring: All vertices have same list of possible colors.
- List coloring: Every vertex has it's own list of possible colors $L(v)$.

Definition

The list coloring is assignment colors to the vertices from their own lists. Formally $c : v \to L(v)$

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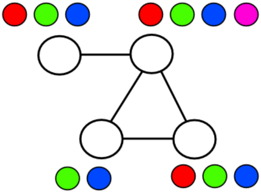

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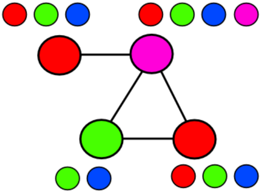

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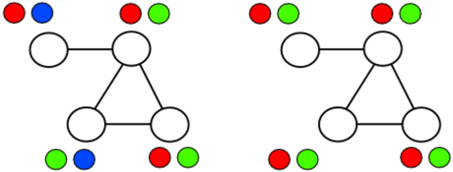
k-Choosable And Choosability

Definition

The graph is *k-choosable* if: Size of every color list is $\geq k$ $\rightarrow$ there is a proprer list coloring.



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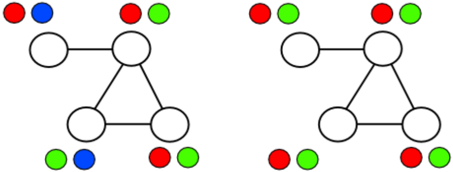

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Rleationship Between Chromatic Number And Choosability

- $\chi(G) \leq \chi_\ell(G)$
- $\chi(G) \leq \Delta(G) + 1$ and also $\chi_\ell(G) \leq \Delta(G) + 1$
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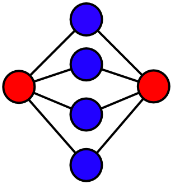

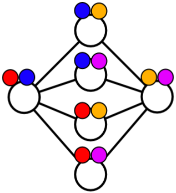

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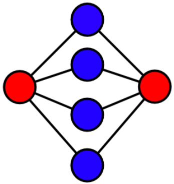

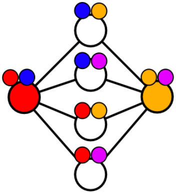

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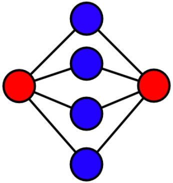

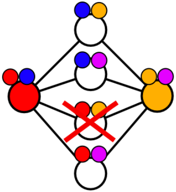

What Is Know For Planar Graphs

Known Theorems:

- Every planar graph is 5-choosable.
  (all cycles)
- Every planar graph without triangles is 4-choosable.
  (no 3)
- Every planar bipartite graph is 3-choosable.
  (no 3, 5, 7, 9, 11, ...)
- There is a non 4-choosable planar graph without triangles.

Problem

Which planar graphs without triangles are 3-choosable?

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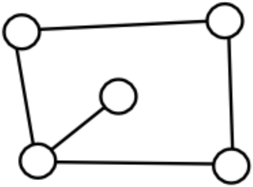
The Idea Of Discharging

- Take an imaginary planar counterexample.
- Remove reducible pieces while keepeing the planarity.
- Assign weights to vertices and faces.
- Move weights if needed and make all weights ≥ 0.
- So the reduced graph is not planar since for all planar graphs holds $\sum$ *weigh* < 0.

Degree Of Vertices And Faces



- Vertex v:
  $\deg v = |\{\text{incident edges}\}|$.
- Face f:
  $\deg f = |\{\text{incident edge sides}\}|$.

$$2|E| = \sum \deg v$$

$$2|E| = \sum \deg f$$

How To Get The Weights

Start from Euler formula for connected graph:

$$|E| = |V| + |F| - 2$$

$$2 * 2|E| + 2|E| = 6|V| + 6|F| - 12$$

$$\sum 2 \deg v + \sum \deg f = 6|V| + 6|F| - 12$$

$$\sum (2 \deg v - 6) + \sum (\deg f - 6) = -12$$

Definition

Weights $w(v) = (2 \deg v - 6), w(f) = (\deg f - 6)$

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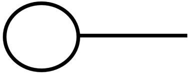
Discharging Application

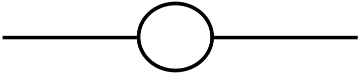
Theorem (1)

Every planar graph without triangles, 4-cycles and 5-cycles is 3-choosable.

The Reduction Part

Removing things without any effect for 3-choosability.





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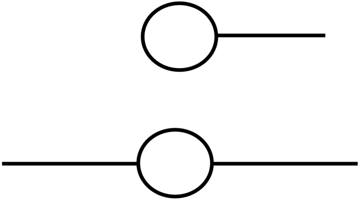

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Counting Weights

deg(?)	$w(v)$	$w(f)$
1	-4	
2	-2	
3	0	-3
4	2	-2
5	4	-1
6	6	0

- $\deg(v) \geq 3 \rightarrow w(v) \geq 0$
- $\deg(f) \geq 6 \rightarrow w(f) \geq 0$

All weights are non-negative.

$$\sum w(v) + \sum w(f) \geq 0$$

But for planar graph must hold

$$\sum w(v) + \sum w(f) = -12$$

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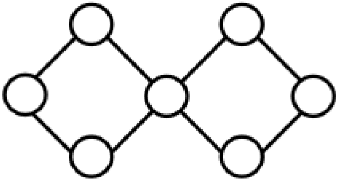
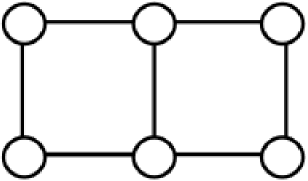
Discharging Application

Theorem (2)

Every planar graph without triangles, 5-cycles and adjacent 4-cycles is 3-choosable.





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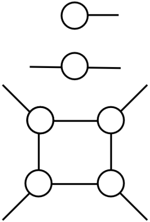

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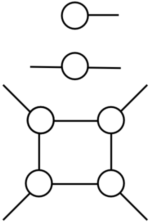

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We have problems with 4-faces. The weight is -2 .

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Dealing with 4-faces

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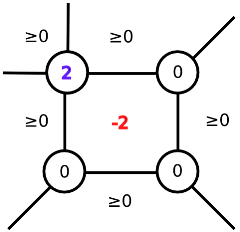

- Every 4-face f has it's own vertex v with $deg(v) \geq 4$ and $w(v) \geq 2$.
- Reassign weights:
  $w'(v) = w(v) - 2$
  $w'(f) = w(f) + 2$.
- So $w'(f) \geq 0$ and $w'(v) \geq 0$ and sum of all weights is same.

For our graph holds $\sum w(v) + \sum w(f) \geq 0$
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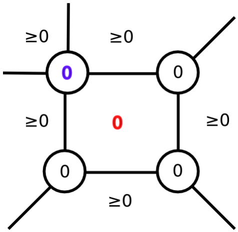

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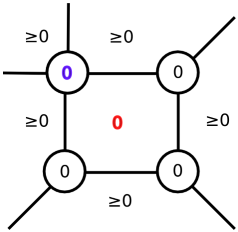

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Summary

- We introduced the list coloring as a generalization of graph coloring.
- We described basics of the discharging method.
- We proved an example from list coloring.

Open Problems

Problem

Is there a non 3-choosable graph without triangles and 5 cycles?

Problem

What if we allow 4 cycles to share a vertex but not edge?

The End

Questions?