

# The $k$ -in-a-path problem for claw-free graphs

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22.1.2010 - CSASC 2010

## Motivation

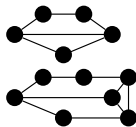
3-IN-A-TREE: Find an induced tree containing given 3 vertices.

Theorem (Chudnovsky, Seymour, to appear)

*The 3-IN-A-TREE problem is solvable in polynomial time.*

Algorithmic consequences and generalizations

- detecting thetas
- detecting pyramids
- 4-IN-A-TREE in triangle-free graphs  
[Derhy et. al. '09]
- $k$ -IN-A-TREE in graphs of girth  $k$   
[Trotignon and Wei '1X]



...  
We:  $k$ -IN-A-TREE for claw-free graphs

3 in a Tree - Molly Monkey

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# Molly by John Rosano monkey

*Farmward Bound*

Front Cover / Summary

Look Inside

About the Author

Questions & Answers

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Upcoming Events

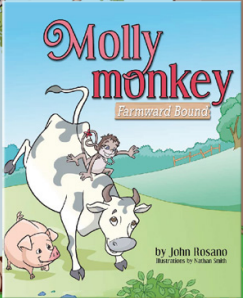
Puzzle Game

Attention Media

Resellers / Wholesale

Molly Monkey loves the jungle life, but she discovers there is much more to see! One afternoon, Molly finds the opportunity of a lifetime. This extraordinary monkey leaves her jungle friends and boards a flight bound for new experiences and exciting adventures. Join Molly in her first adventure, she would love to have you tag along. When you purchase this book, a tree will be planted somewhere in America through American Forests.

\$14.95  
**BUY NOW**  
click to purchase this book.



by John Rosano  
Illustrations by Nathan Seab

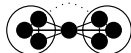
This website designed by the helpful folks at [Dragonpencil.com](http://Dragonpencil.com) Use your mouse to flip the page corners.

<http://www.3inatree.net/>

## Definitions - quick reminder

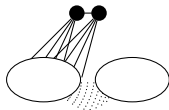
A graph  $G$  is

- *claw-free* - no induced claw
- *quasi-line* -  $N(v)$  is union of two cliques
- a *line graph* -  $G = L(H)$  for some  $H$
- an *interval graph*



A graph  $G$  has

- a *k-hole* - induced  $k$ -cycle ( $k \geq 4$ )
- an *anti-hole* - induced complement of  $k$ -cycle
- a *homogeneous clique*



## $k$ -IN-A-PATH for claw-free graph

- $k$ -IN-A-PATH: Find an induced path containing given  $k$  vertices (terminals).
- $k$ -IN-A-TREE is  $k$ -IN-A-PATH for claw-free graphs



### Theorem

$k$ -IN-A-PATH is solvable in polynomial time for claw-free graphs.  
( $k$  constant)

### Theorem

$k$ -IN-A-PATH is NP-complete if  $k$  is part of the input for line graphs.

## Algorithm overview

### Theorem

$k$ -IN-A-PATH is solvable in polynomial time for claw-free graphs.  
( $k$  constant)

claw-free  $G$ , terminals  $T$ ,  $|T| = k$ , find induced  $P$  s.t.  $T \subseteq P$

- fix the path a bit
- make  $G$  quasi-line
- make  $G$  quasi-line with no homogeneous clique
- make  $G$  circular interval or composition of interval graphs
- solve circular interval graph
- or solve intervals and  $k$ -DISJOINT-PATHS for a line graph

## Algorithm overview - the path

claw-free  $G$ , terminals  $T$ ,  $|T| = k$ , find induced  $P$  s.t.  $T \subseteq P$

- fix the path a bit
  - $k \geq 3$
  - terminals are ordered  $t_1, t_2, \dots, t_k$
  - terminals and their neighbour are of degree  $\leq 2$
- make  $G$  quasi-line
- make  $G$  quasi-line with no homogeneous clique
- make  $G$  circular interval or composition of interval graphs
- solve circular interval graph
- or solve intervals and  $k$ -DISJOINT-PATHS for a line graph

## Algorithm overview - quasi-line

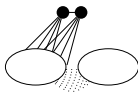


claw-free  $G$ , terminals  $T$ ,  $|T| = k$ , find induced  $P$  s.t.  $T \subseteq P$

- fix the path a bit
- make  $G$  quasi-line
  - clean  $G$  (no odd  $\geq 7$ -anti-hole) [Hof, Kamiński, Paulusma '09]
  - remove vertices which have 5-anti-hole in neighbourhood
  - result is quasi-line as no odd anti-hole among neighbour
- make  $G$  quasi-line with no homogeneous clique
- make  $G$  circular interval or composition of interval graphs
- solve circular interval graph
- or solve intervals and  $k$ -DISJOINT-PATHS for a line graph



## Algorithm overview - no homogeneous clique



claw-free  $G$ , terminals  $T$ ,  $|T| = k$ , find induced  $P$  s.t.  $T \subseteq P$

- fix the path a bit
- make  $G$  quasi-line
- make  $G$  quasi-line with no homogeneous clique
  - easy to check if edge is a homogeneous clique
  - contract homogeneous edge
- make  $G$  circular interval or composition of interval graphs
- solve circular interval graph
- or solve intervals and  $k$ -DISJOINT-PATHS for a line graph

## Algorithm overview - intervals

claw-free  $G$ , terminals  $T$ ,  $|T| = k$ , find induced  $P$  s.t.  $T \subseteq P$

- fix the path a bit
- make  $G$  quasi-line
- make  $G$  quasi-line with no homogeneous clique
- make  $G$  circular interval or composition of interval graphs
  - find all homogeneous pairs of cliques [King, Reed '08] and contract them
  - result is circular interval or the composition [Chudnovsky, Seymour '05]
  - decide circular or composition [Deng, Hell, Huang '96]
- solve circular interval graph
- or solve intervals and  $k$ -DISJOINT-PATHS for a line graph

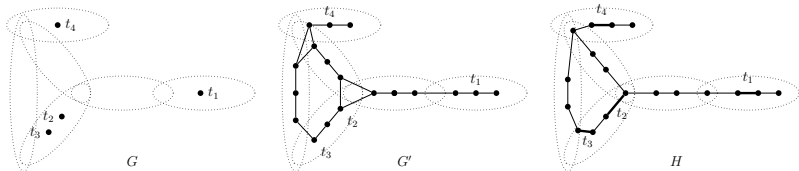
## Algorithm overview - circular interval

claw-free  $G$ , terminals  $T$ ,  $|T| = k$ , find induced  $P$  s.t.  $T \subseteq P$

- fix the path a bit
- make  $G$  quasi-line
- make  $G$  quasi-line with no homogeneous clique
- make  $G$  circular interval or composition of interval graphs
- solve circular interval graph
  - get circular representation [Deng, Hell, Huang '96]
  - solve
- or solve intervals and  $k$ -DISJOINT-PATHS for a line graph

## Algorithm overview - composition of interval graphs

- find the composition [King, Reed '08]
- solve each interval graph separately
- replace strips by short paths -  $G'$



- get a graph  $H$  such that  $G' = L(H)$
- get an instance of  $k$ -DISJOINT-PATHS on  $H$
- solve  $k$ -DISJOINT-PATHS [Robertson, Seymour '95]

## Corollaries

### Theorem

*The following problems are polynomial time solvable on claw-free graphs for a fixed  $k$ :*

- $k$ -INDUCED DISJOINT PATHS
- $k$ -INDUCED CYCLE
- 2 MUTUALLY INDUCED HOLES

## Open problems

Determine the computational complexity for

- ODD HOLE
- 2 MUTUALLY INDUCED HOLES

both are polynomial time solvable for claw free graphs  
mutually induced odd holes is NP-complete