Extending 3-coloring of a face in triangle-free planar graphs

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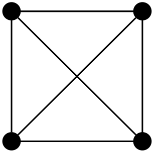
Definitions (critical graphs)

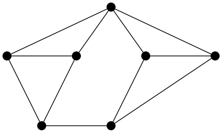
graph $G = (V, E)$

coloring is $\varphi : V \to K$ such that $\varphi(u) \neq \varphi(v)$ if $uv \in E$

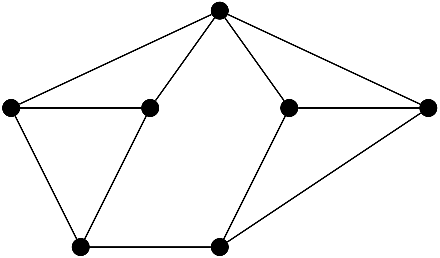
G is a *k-colorable* if coloring with $|K| = k$ exists

G is a *k-critical graph* if G is not $(k-1)$-colorable
but every $H \subset G$ is $(k-1)$-colorable.





Cuts in a 4-critical graph G



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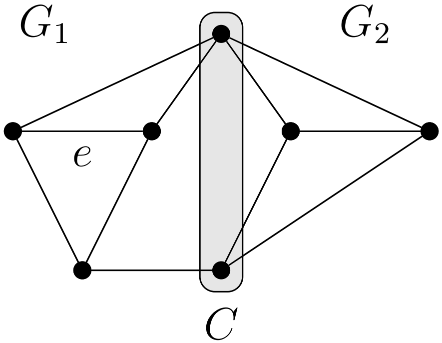
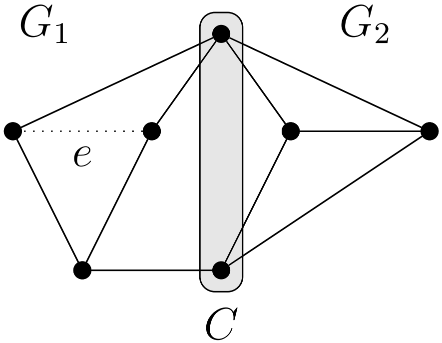


Cuts in a 4-critical graph *G*



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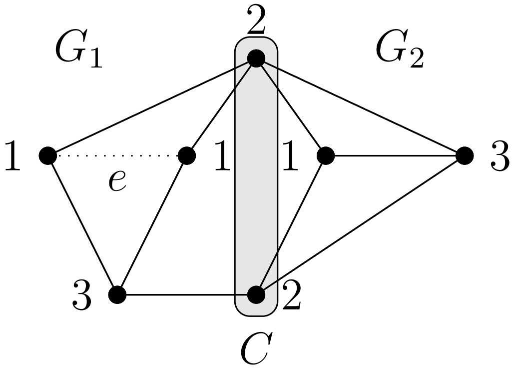
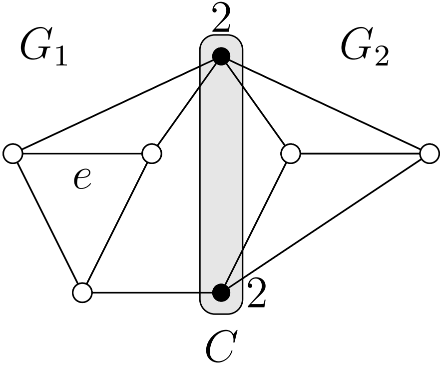


Cuts in a 4-critical graph G

Observation

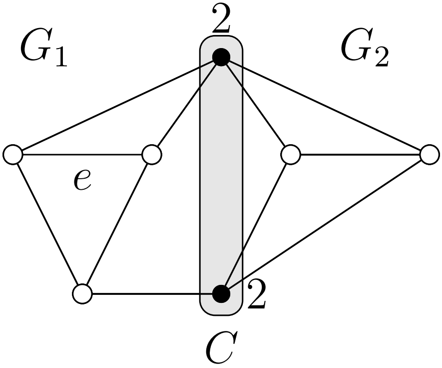
There exists a 3-coloring of $V(C)$ that extends to $G_1 - e$ but does not extend to G_1.



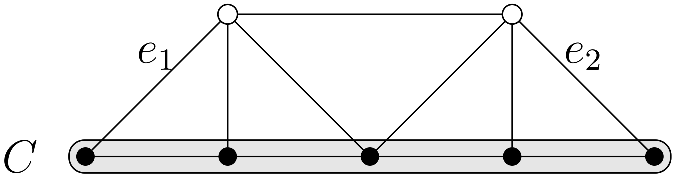
Cuts in a 4-critical graph G

Observation

For every cut C and every $e \in V(G_1)$ exists a 3-coloring of $V(C)$ that extends to $G_1 - e$ but does not extend to G_1.



Different colorings for different edges



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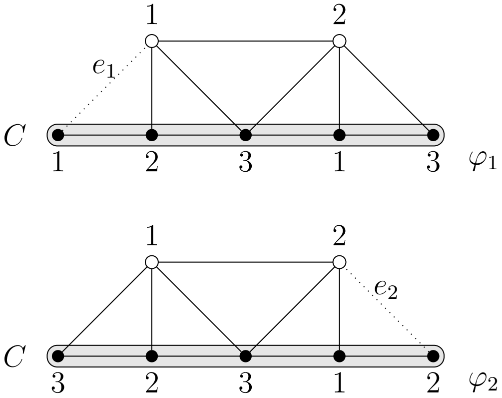

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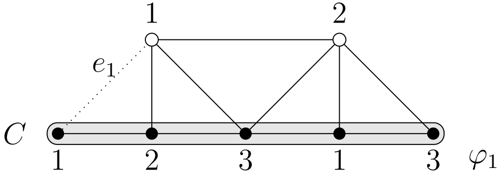
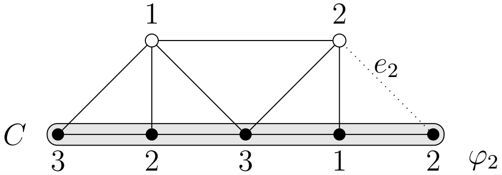
A graph G is C-*critical* for k-coloring if for every $e \in E(G)$ exists a k-coloring φ_e of $V(C)$ that extends to $G - e$ but does not extend to G.





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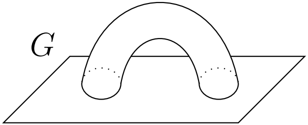
If G *is* $(k+1)$*-critical, then* G *is* $\emptyset$*-critical for* k*-coloring.*

Which C for C-critical?

- simplifying graphs on surfaces



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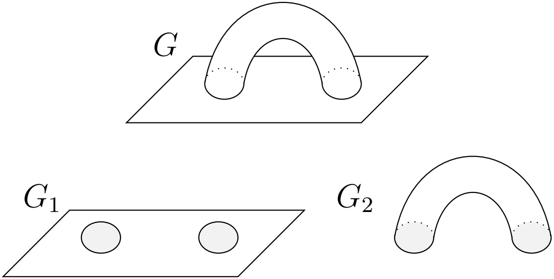

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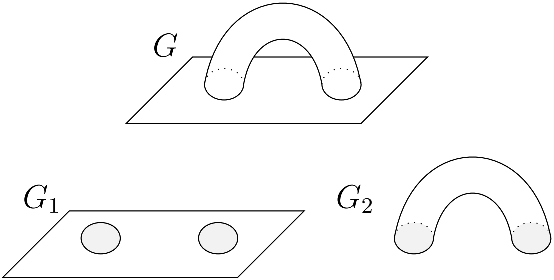

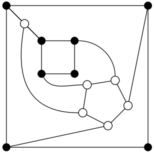

G_2

Which C for C-critical?

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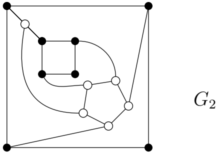
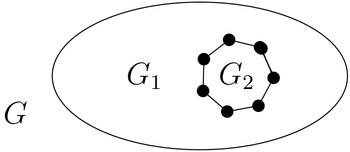


- interior of a cycle



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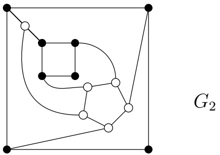

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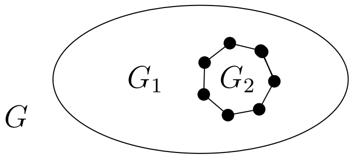

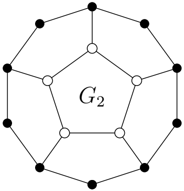

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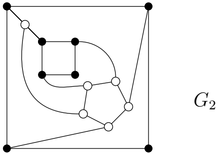

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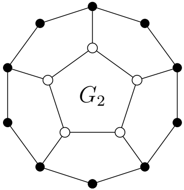
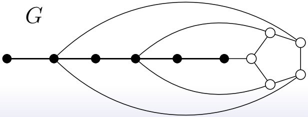


- precolored tree



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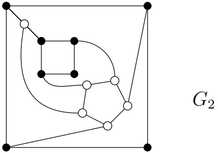

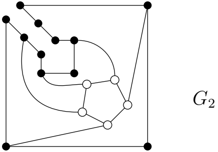

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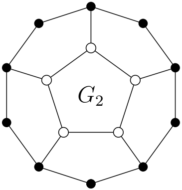

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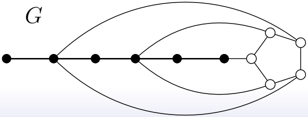

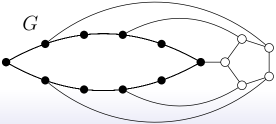

Our focus

We focus on G that is

- plane
- outer-face is a cycle C
- G is C-critical for 3-coloring

Goal: For a given length of C enumerate all C-critical graphs.

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Known results (girth 5)

C-critical plane graphs of girth 5 are precisely enumerated for

- $|C| \leq 11$ by Thomassen '03 and Walls '99
- $|C| = 12$ by Dvořák and Kawarabayashi '11
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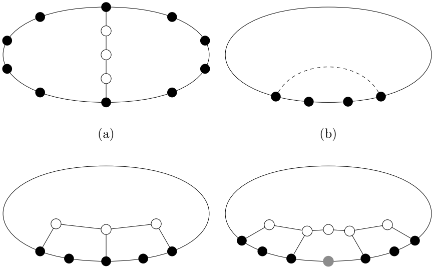
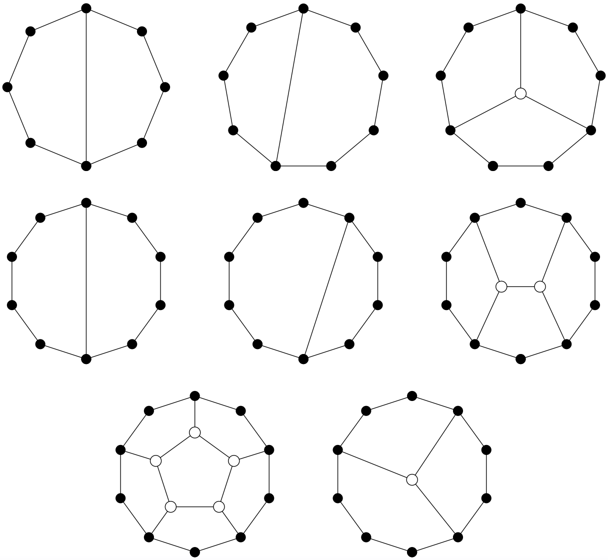
Recursive description for all $|C|$ by Dvořák and Kawarabayashi '11



(a) (b) (c) (d)

$|C| \leq 10$ (girth 5)



Known results (girth 4)

No recursive enumeration for girth 4 know.

C-critical plane graphs of girth 4 precisely enumerated for

- $|C| \in \{4, 5\}$ by Aksenov '74
- $|C| = 6$ by Gimbel and Thomassen '97
- $|C| = 6$ by Aksenov, Borodin, and Glebov '03
- $|C| = 7$ by Aksenov, Borodin, and Glebov '04
- $|C| = 8$ by Dvořák and L. '13+
- $|C| = 9$ by Choi, Ekstein, Holub, and L. (in writing)

$|C| \in \{4, 5, 6\}$ (girth 4)

Theorem (Aksenov '74)

If G is a plane graph of girth 4, then every pre-coloring of C_4 and C_5 extends to G.

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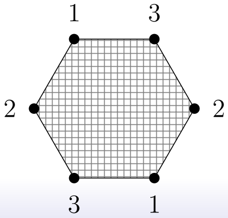
Theorem (Gimbel and Thomassen '97; Aksenov, Borodin, and Glebov '03)

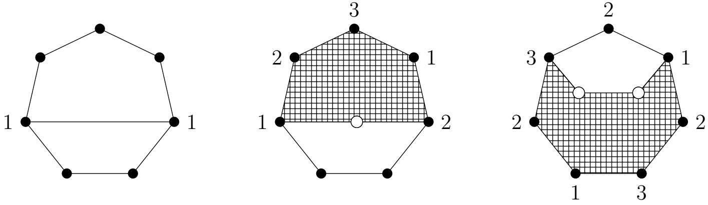
Let G be a plane triangle-free graph with chordless outer 6-cycle C. G is C-critical if and only if G contains no separating 4-cycles and all other faces of G are 4-faces (i.e. G is a quadrangulation). Moreover, a 3-coloring of C does not extend to G if and only if opposite vertices of C are colored the same.



$|C| = 7$ (girth 4)

Theorem (Aksenov, Borodin, and Glebov '04)

If G is a plane triangle-free graph with outer face bounded by a cycle C of length 7 then G is C-critical iff G looks like (a), (b), or (c).

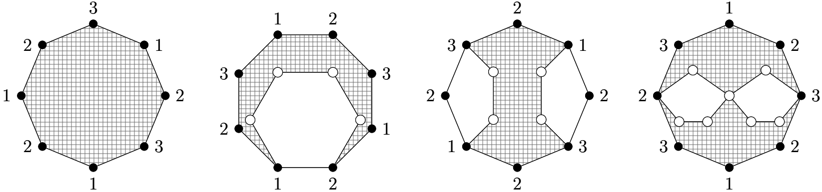


(a) (b) (c)

$|C| = 8$ (girth 4)

Theorem (Dvořák and L.)

If G is a plane triangle-free graph with outer face bounded by a cycle C of length 8 then G is C-critical iff G looks like (a), (b), (c), or (d).



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Tool

Theorem (Tutte '54)

A plane graph G has a 3-coloring iff its dual $G^\star$ has a nowhere-zero $\mathbb{Z}_3$-flow.

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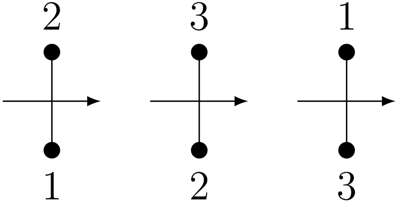

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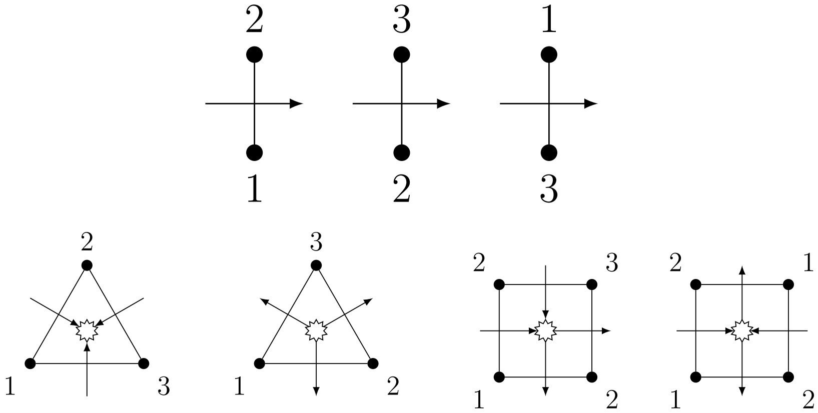
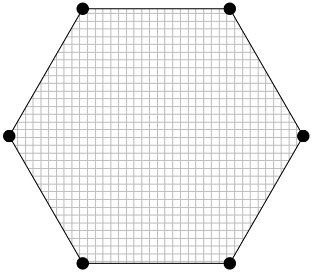


C-critical quadrangulation



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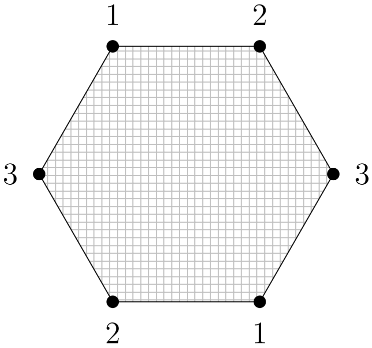

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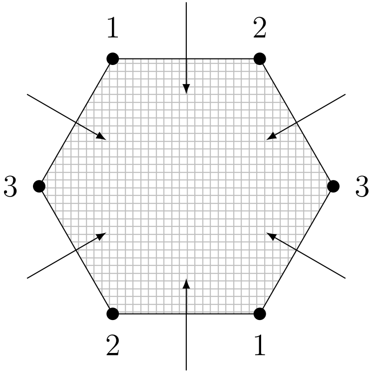

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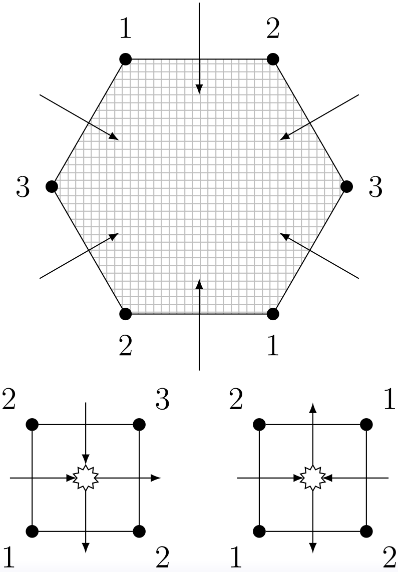

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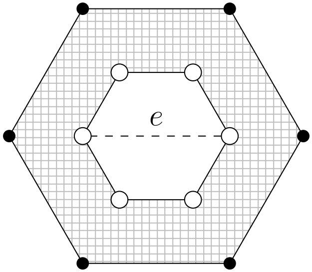

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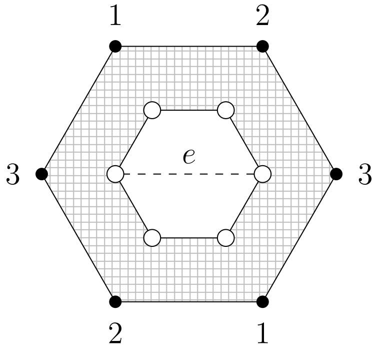

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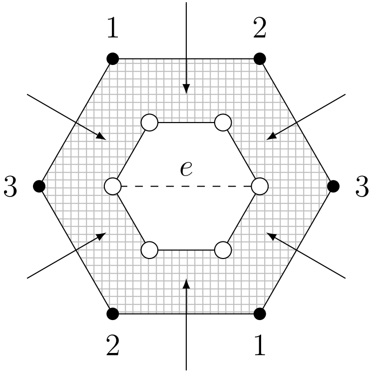

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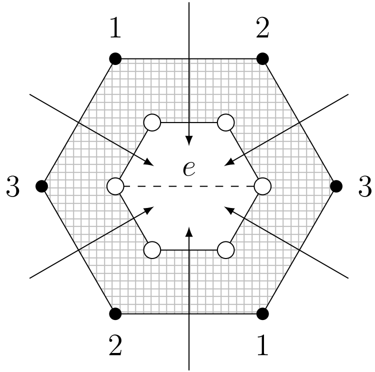

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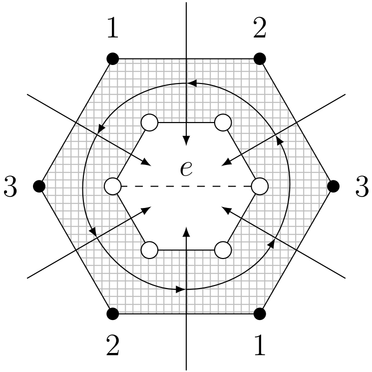

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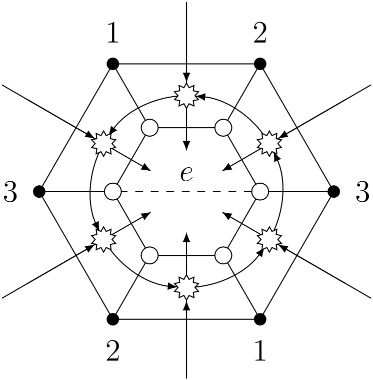

$|C| = 8$

Corollary (Dvořák, Kráľ, Thomas)

If G is a C-critical, plane, triangle-free graph, where $|C| = 8$, then

$$\{\emptyset, \{7\}, \{5, 5\}\}$$

are the only possible multisets of face lengths ≥ 5.

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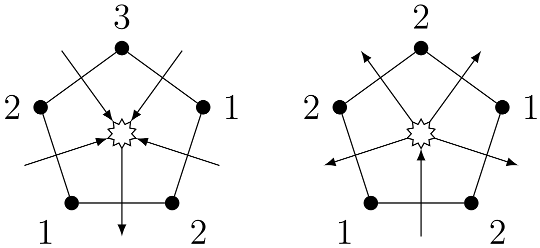
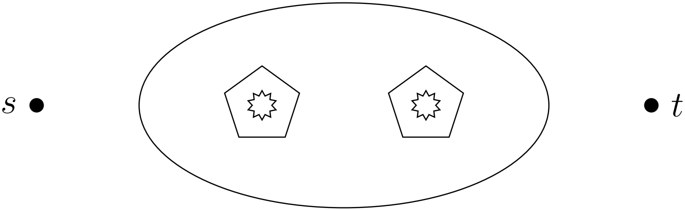


$|C| = 8$, two 5-faces

Finding G and a coloring of C that does not extend.
"source edges = sink edges"



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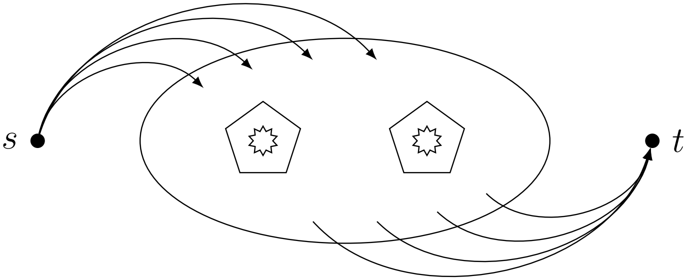

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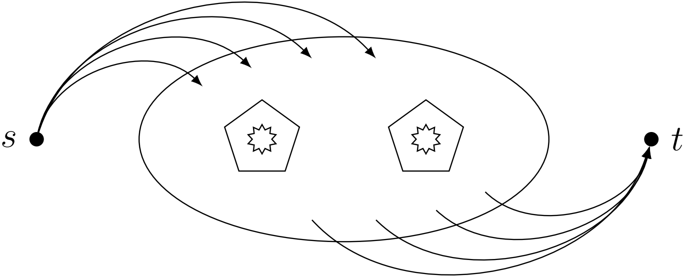

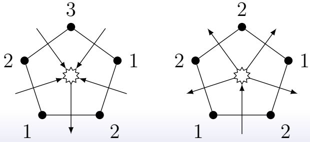

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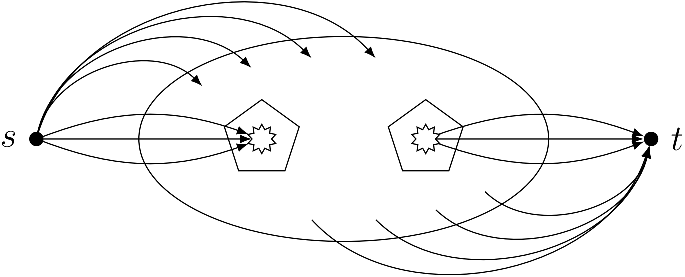

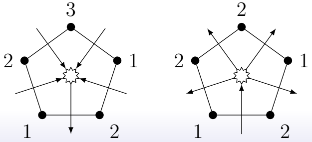

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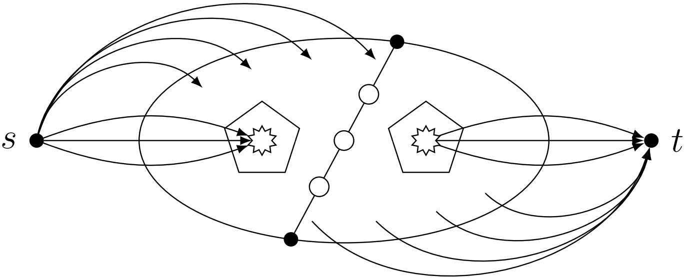

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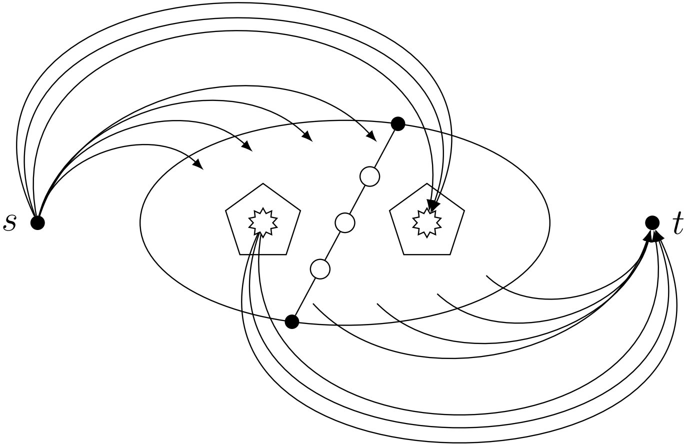

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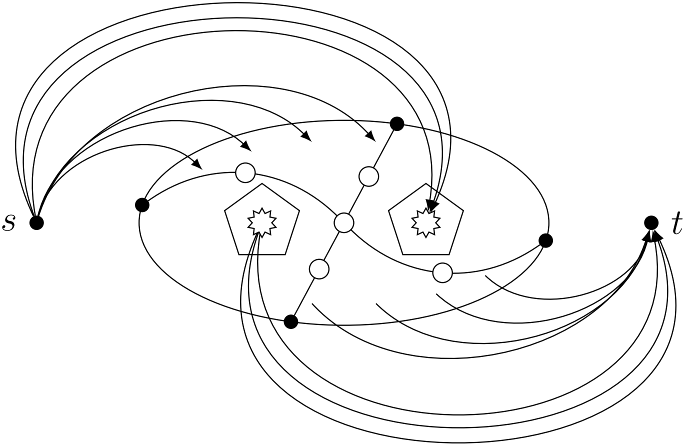

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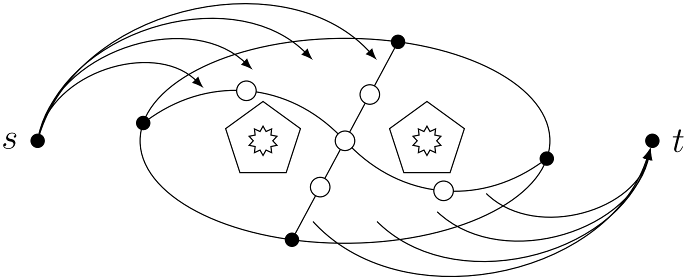
Finding G and a coloring of C that does not extend.

"source edges = sink edges"



Thank you for your attention!