Rainbow triangles in 3-edge-colored graphs

Bernard Lidický

Iowa State University

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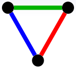
Sep 21, 2014

joint work with
József Balogh, Ping Hu, Florian Pfender, Jan Volec, Michael Young

The problem

$F(n) :=$ max  over all 3-edge-colorings of K_n

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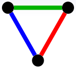
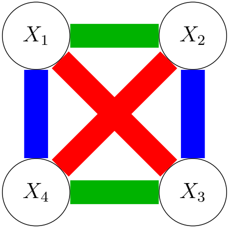
$F(n) :=$ max  over all 3-edge-colorings of K_n



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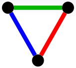
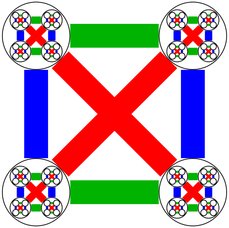
$F(n) :=$ max  over all 3-edge-colorings of K_n



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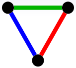
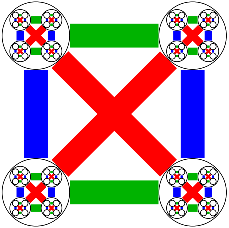
$F(n) :=$ max  over all 3-edge-colorings of K_n



Conjecture (Erdős, Sós)

This construction is the best possible. In other words,

$$F(n) = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 + \sum_i F(x_i),$$

where $x_1 + x_2 + x_3 + x_4 = n$, *and* $|x_i - x_j| \leq 1$.

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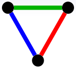
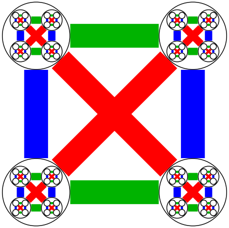
$F(n) :=$ max  over all 3-edge-colorings of K_n



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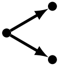
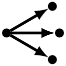
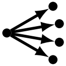
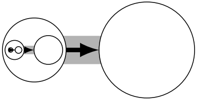
This talk: The conjecture is true for n large and for any $n = 4^k$.

Known extremal problems with iterated constructions

Theorem (Falgas-Ravry, Vaughan)

Density of  *,*  *and*  *is maximized by*  *.*

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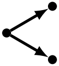
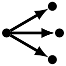
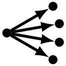
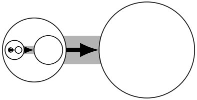
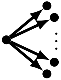
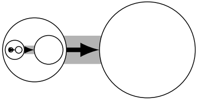
Density of  ,  *and*  *is maximized by*  .

Theorem (Huang)

Density of  *is maximized by*  .

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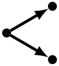
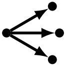
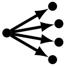
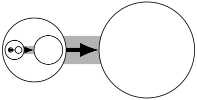
Density of  ,  *and*  *is maximized by*  .

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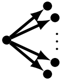
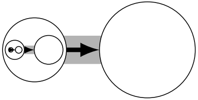
Density of  *is maximized by*  .

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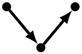
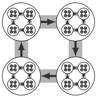
Density of  *is maximized by*  .

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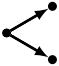
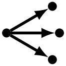
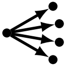
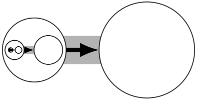
Density of  ,  *and*  *is maximized by*  .

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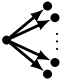
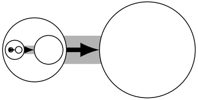
Density of  *is maximized by*  .

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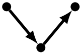
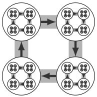
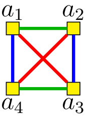
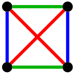
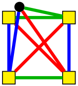
Density of  *is maximized by*  .

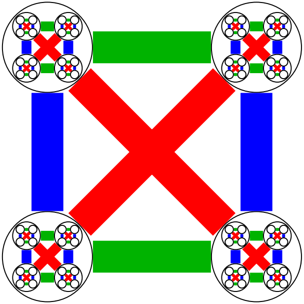
Theorem (Pikhurko)

Iterated blow-up of r-graph is extremal for $\pi(\mathcal{F})$ for some family $\mathcal{F}$.

Proof strategy for n large

- pick 

typical copy of  , classify vertices via 



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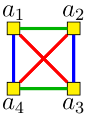
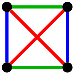
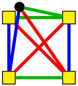
- pick 

typical copy of  , classify vertices via 
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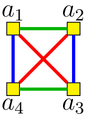
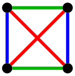
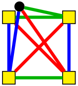
- pick 

typical copy of  , classify vertices via 
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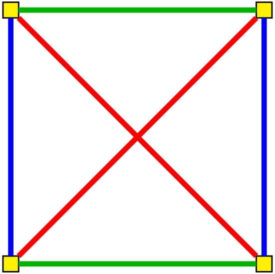

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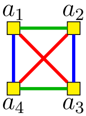
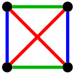
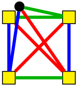
- pick 

typical copy of  , classify vertices via 
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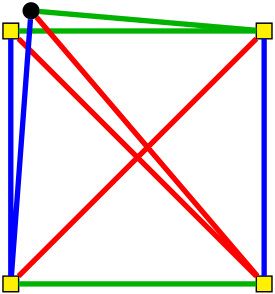

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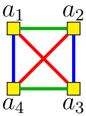
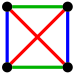
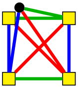
- pick 

typical copy of , classify vertices via 
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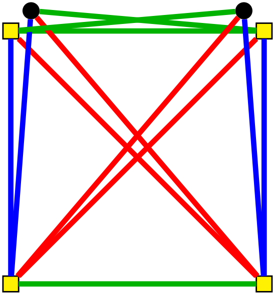

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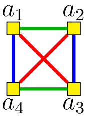
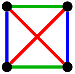
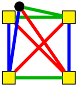
- pick 

typical copy of  , classify vertices via 

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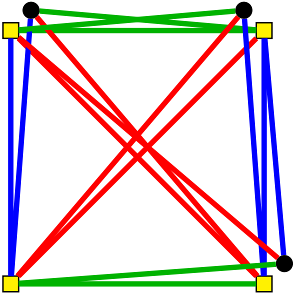

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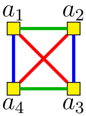
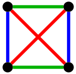
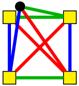
- pick  typical copy of , classify vertices via 

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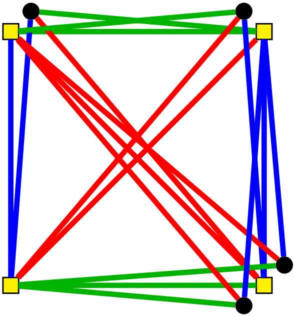

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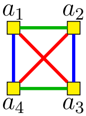
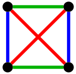
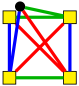
- pick 

typical copy of , classify vertices via 
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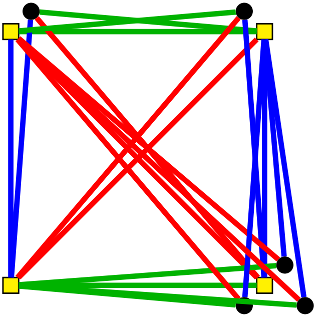

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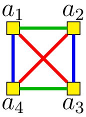
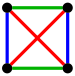
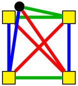
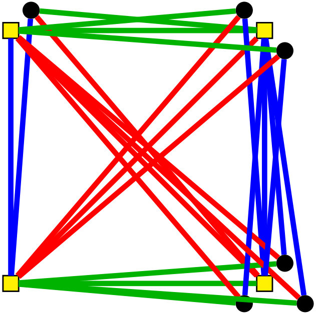
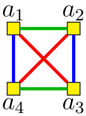
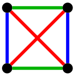
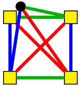
- pick 

typical copy of  , classify vertices via 
- $X_i :=$ clones of a_i $\qquad T := V \setminus \cup X_i \qquad W$ wrong $X_i \leftrightarrow X_j$ edges

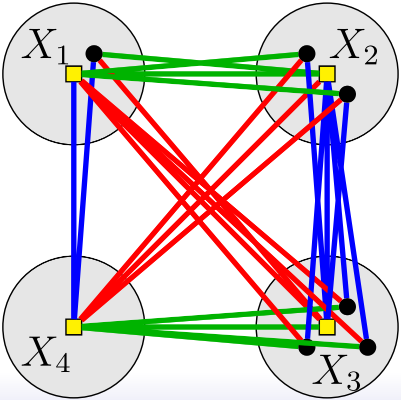


Proof strategy for n large

- pick 

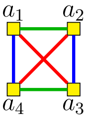
typical copy of  , classify vertices via 

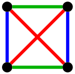
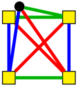
- $\mathbf{X_i}$ **:= clones of** $\mathbf{a_i}$ $\quad T := V \setminus \cup X_i \quad W$ wrong $X_i \leftrightarrow X_j$ edges



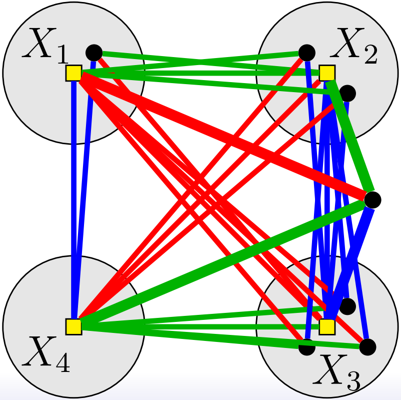
- parts $|X_i| = (1/4 \pm 10^{-2}) \cdot n$

Proof strategy for n large

- pick 

typical copy of  , classify vertices via 

- $X_i :=$ clones of a_i $\qquad T := V \setminus \cup X_i$ $\qquad W$ wrong $X_i \leftrightarrow X_j$ edges



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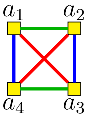
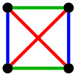
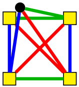
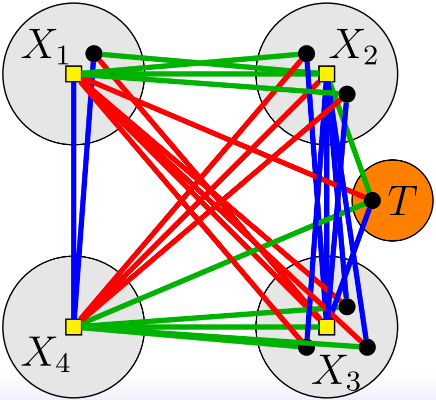
Proof strategy for n large

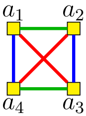
- pick 

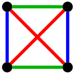
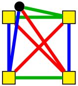
typical copy of , classify vertices via 
- $X_i :=$ clones of a_i $\quad$ $\mathbf{T} := \mathbf{V} \setminus \cup \mathbf{X_i}$ $\quad$ W wrong $X_i \leftrightarrow X_j$ edges



- parts $|X_i| = (1/4 \pm 10^{-2}) \cdot n$
- trash $|T| < 10^{-2} \cdot n$

Proof strategy for n large

- pick 

typical copy of  , classify vertices via 

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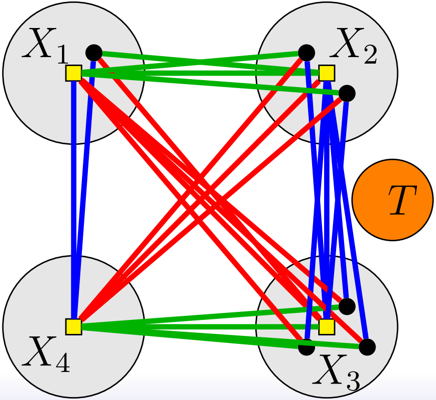

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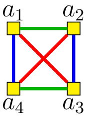
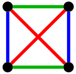
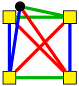
Proof strategy for n large

- pick 

typical copy of  , classify vertices via 

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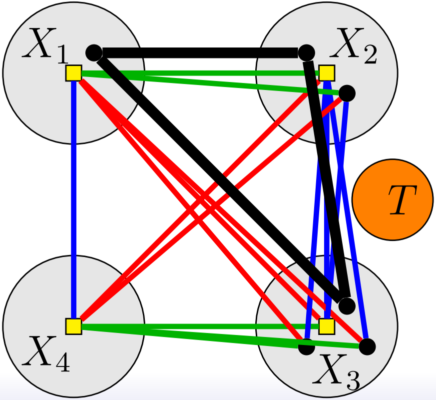

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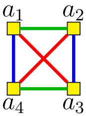
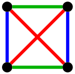
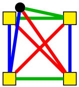
Proof strategy for n large

- pick 

typical copy of  , classify vertices via 
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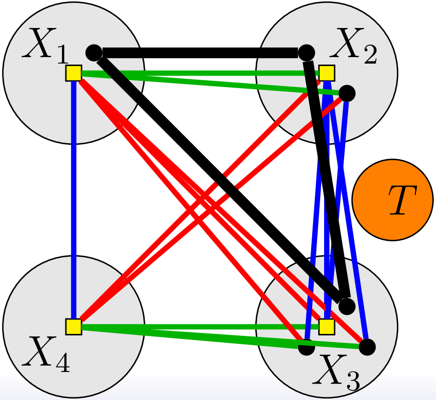

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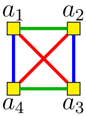
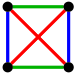
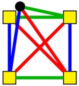
Proof strategy for n large

- pick 

typical copy of , classify vertices via 
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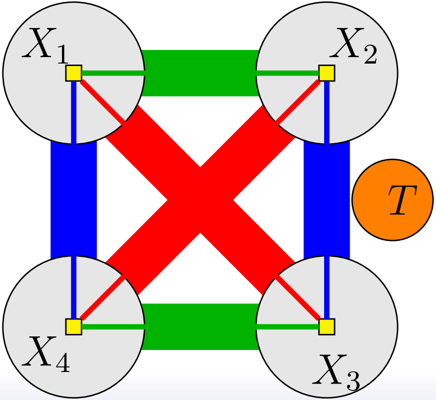

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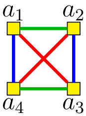
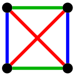
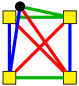
- pick 

typical copy of , classify vertices via 
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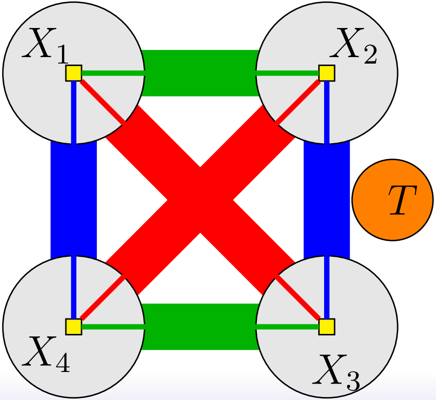

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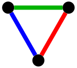
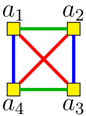
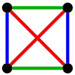
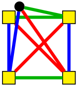
- vertex in T is in few 

Proof strategy for n large

- pick 

typical copy of  , classify vertices via 

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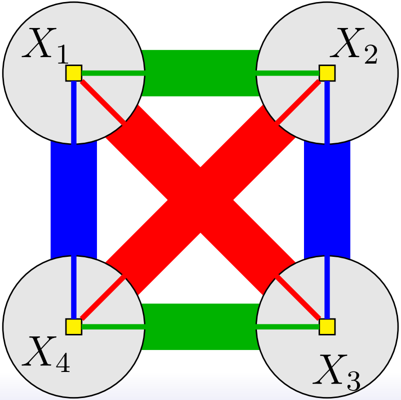

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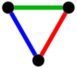
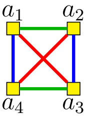
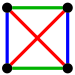
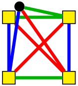
- vertex in T is in few 

Proof strategy for n large

- pick 

typical copy of  , classify vertices via 
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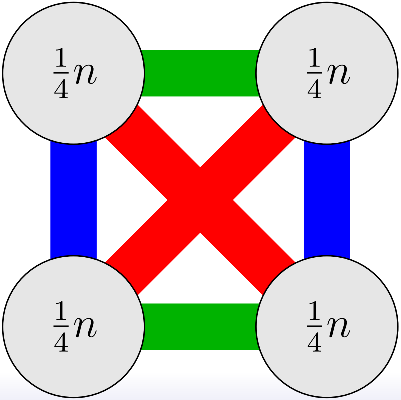

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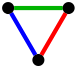
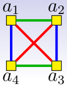
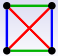
- vertex in T is in few 
- show that the parts are balanced

Proof strategy: 

typical copy of 

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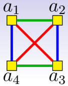
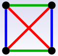
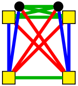
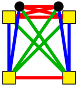
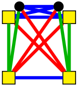
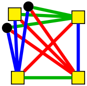
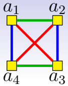
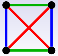
Proof strategy: 

typical copy of 

- X_i := clones of a_i $\quad$ $T := V \setminus \cup X_i$ $\quad$ W wrong $X_i \leftrightarrow X_j$ edges
- Goal: $\alpha \cdot \Big($  $+$  $+$  $\Big) - \beta \cdot$  $\geq \delta$ for α, β, δ

Proof strategy:  typical copy of 

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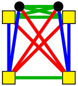
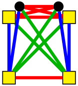
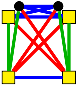
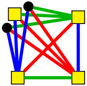
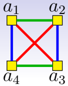
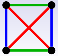
- Goal: $\alpha \cdot \Big($  $+$  $+$  $\Big) - \beta \cdot$  $\geq \delta$ for α, β, δ
- $\Longrightarrow \alpha \cdot \left(\sum_{i \neq j} |X_i||X_j| - 2|W| \right) - \beta \cdot \sum_i |X_i|^2 \geq \delta \cdot n^2$

Proof strategy: 

typical copy of 

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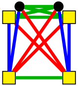
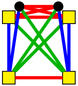
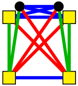
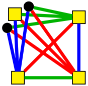
- Goal: $1 \cdot \Big($  $+$  $+$  $\Big) - \frac{26}{9} \cdot$  $\geq \frac{27}{1000}$
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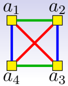
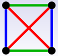
Proof strategy: 

typical copy of 

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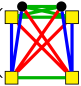
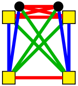
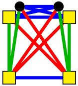
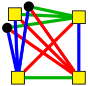
- Goal: $1 \cdot \Big($  $+$  $+$  $\Big) - \frac{26}{9} \cdot$  $\geq \frac{27}{1000}$
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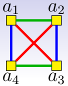
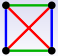
Proof strategy:  typical copy of 

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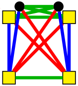
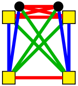
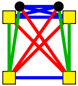
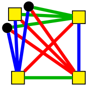
- Goal: $1 \cdot \Big($  $+$  $+$  $\Big) - \frac{26}{9} \cdot$  $\geq \frac{27}{1000}$
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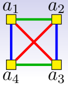
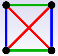
Proof strategy: 

typical copy of 

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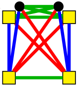
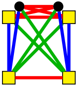
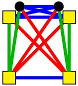
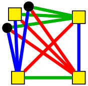
- Goal: $1 \cdot \Big($  $+$  $+$  $\Big) - \frac{26}{9} \cdot$  $\geq \frac{27}{1000}$
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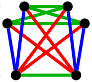
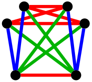
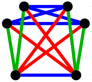
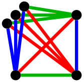
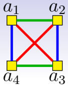
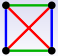
- Claim: $\frac{4}{15} \cdot \Big($  $+$  $+$  $\Big) - \frac{26}{45} \cdot$  $\geq \frac{9}{3500}$

Proof strategy: 

typical copy of 

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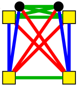
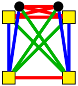
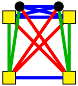
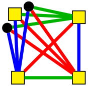
- Goal: $1 \cdot \Big($  $+$  $+$  $\Big) - \frac{26}{9} \cdot$  $\geq \frac{27}{1000}$
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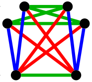
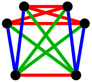
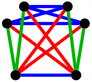
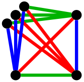
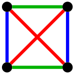
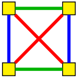
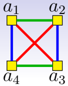
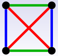
- Claim: $\frac{4}{15} \cdot \Big($  $+$  $+$  $\Big) - \frac{26}{45} \cdot$  $\geq \frac{9}{3500}$
- Claim:  $\approx \frac{2}{21}$. Then average  has value $\frac{9/3500}{2/21} = \frac{27}{1000}$

Proof strategy: 

typical copy of 

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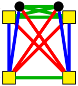
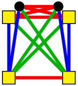
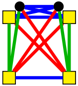
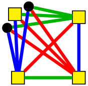
- Goal: $1 \cdot \Big($  $+$  $+$  $\Big) - \frac{26}{9} \cdot$  $\geq \frac{27}{1000}$
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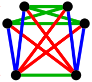
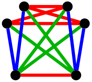
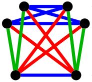
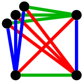
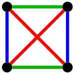
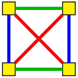
- Claim: $\frac{4}{15} \cdot \Big($  $+$  $+$  $\Big) - \frac{26}{45} \cdot$  $\geq \frac{9}{3500}$
- Claim:  $\approx \frac{2}{21}$. Then average  has value $\frac{9/3500}{2/21} = \frac{27}{1000}$

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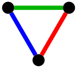
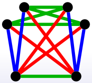
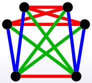
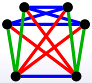
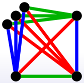
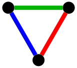
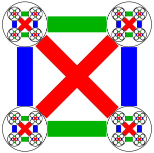
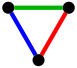
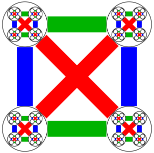
- semidefinite method: relaxing optimization problems on LIM
- we optimize on $\text{LIM}^{\text{EXT}} = \{q \in \text{LIM} : q($  $) \geq 0.4\}$

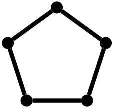
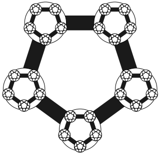
$\min_{q \in \text{LIM}^{\text{EXT}}} q\left(\frac{4}{15} \cdot \left(\right.\right.$  $+$  $+$  $\left.\right) - \frac{26}{45} \cdot$  $\left.\right)$

Conclusion and related problems

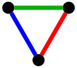
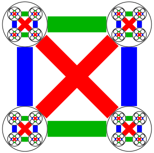
Density of  is maximized by  .

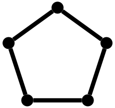
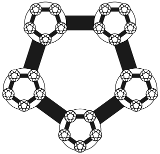
Conclusion and related problems

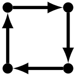
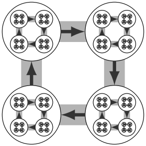
Density of  is maximized by  .

Induced density of  is maximized by  .

Conclusion and related problems

Density of  is maximized by  .

Induced density of  is maximized by  .

Induced density of  is maximized by  .


Thank you for listening!