3-coloring triangle-free planar graphs with a precolored 9-cycle

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IWOCA - Duluth

October 17, 2014

A graph G is k-colorable if there is a function f where

- for each vertex v: $f(v) \in [k]$
- for each edge xy: $f(x) \neq f(y)$

A graph G is k-critical if

- G is not $(k-1)$-colorable
- every proper subgraph H of G is $(k-1)$-colorable

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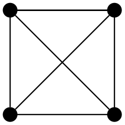
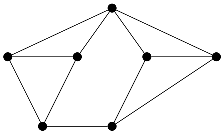
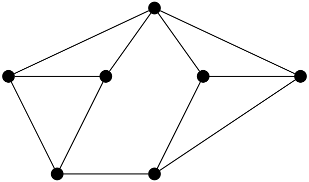




Cuts in critical graphs



4-critical

- not 3-colorable
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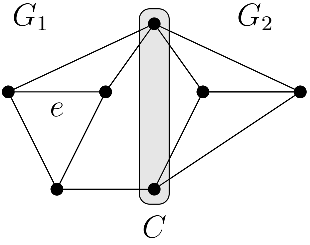

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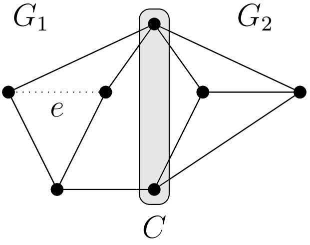

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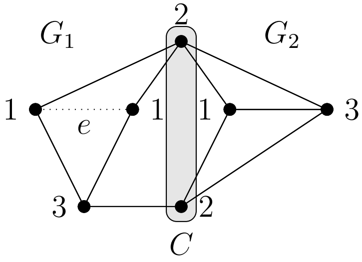

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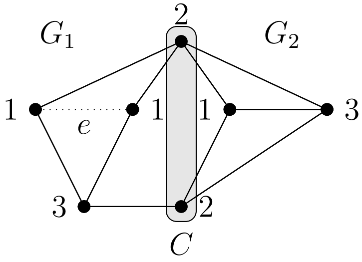
Observation

There exists a 3-coloring of C that extends to $G_1 - e$ but does not extend to G_1.



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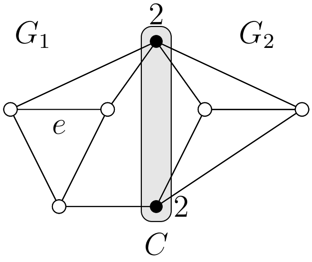

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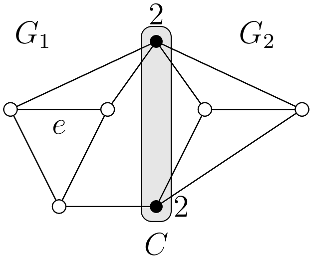

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Main definition

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For a graph G and $C \subseteq V(G)$ we say G is C-*critical* for k-coloring if for each $e \in E(G)$, there exists a k-coloring φ_e of C that extends to $G - e$ but does not extend to G.

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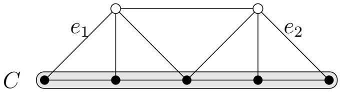

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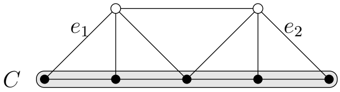

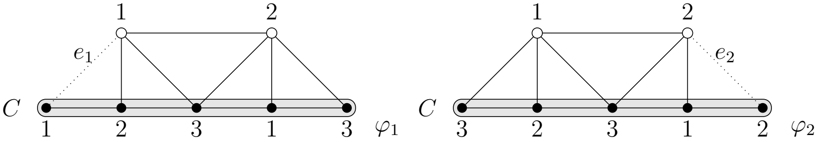

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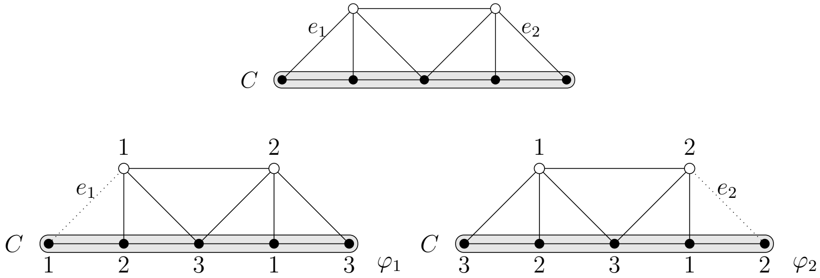

Observation

If G is $(k+1)$-critical, then G is $\emptyset$-critical for k-coloring.

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- simplifying graphs on surfaces

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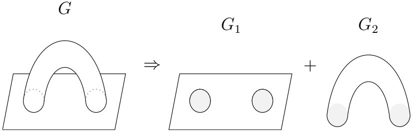

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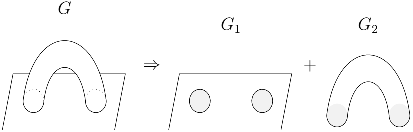

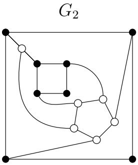

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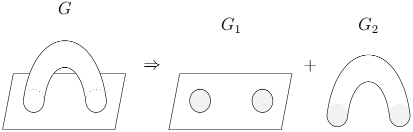

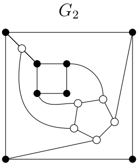

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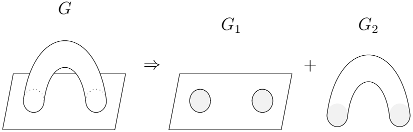

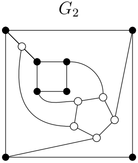

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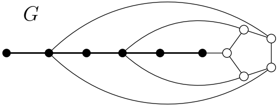

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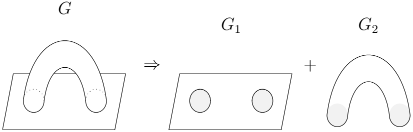

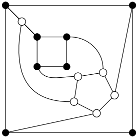

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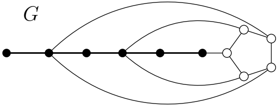

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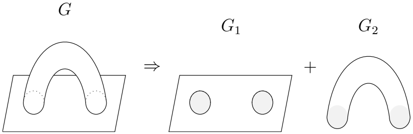

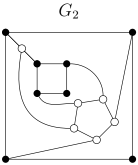

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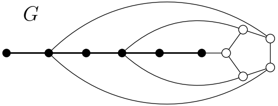

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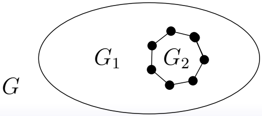

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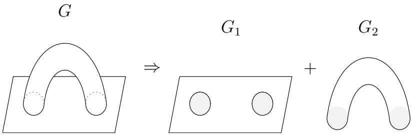

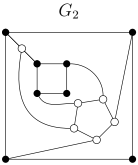

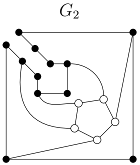

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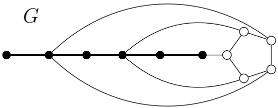

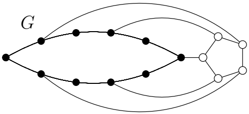

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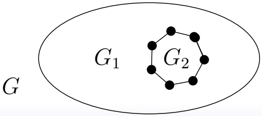

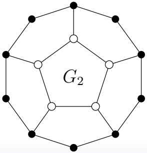

Focus: plane graphs that are C-critical for 3-coloring where C is a cycle.

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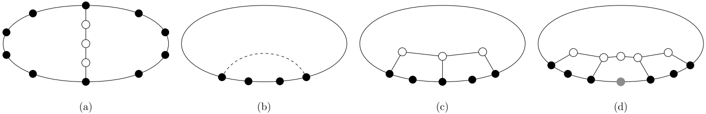
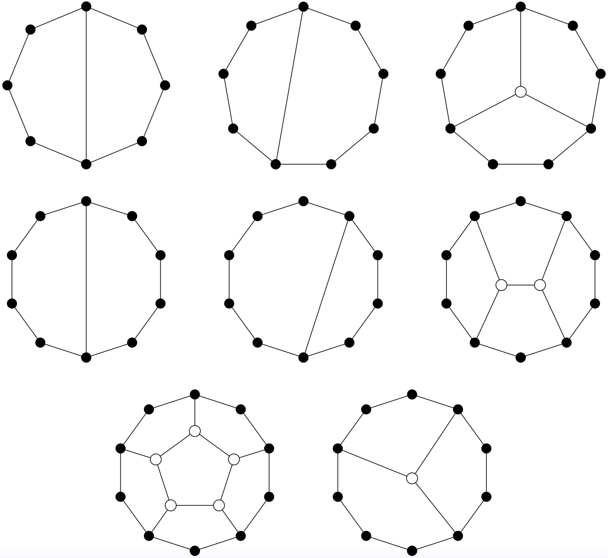
Recursive description for all $|C|$ by Dvořák–Kawarabayashi 2011



C-critical plane graphs of girth 5 with $|C| \leq 10$



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- $|C| = 9$ by Choi–Ekstein–Holub–Lidický 2014+

C-critical for girth 4 are more difficult since there are infinitely many of them of $|C| \geq 6$

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"nice" plane graph: has no separating 4-cycles or 5-cycles.

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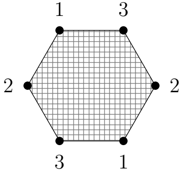

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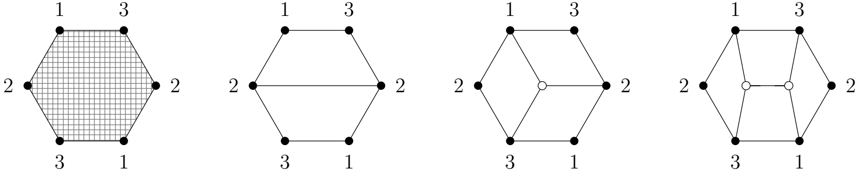

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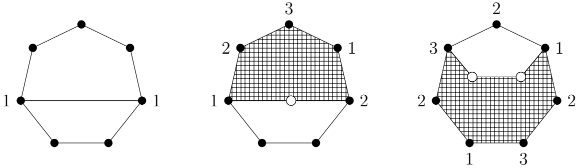

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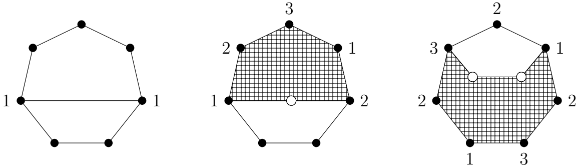
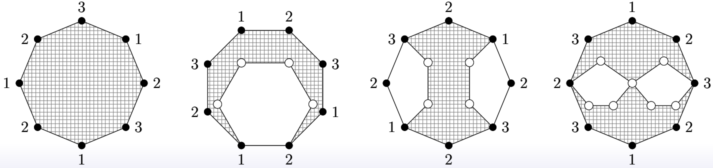


Theorem (Dvořák–Lidický 2013+)

If G is a "nice" plane graph of girth 4 bounded by an 8-cycle C, then G is C-critical if and only if G "looks like" a graph below.



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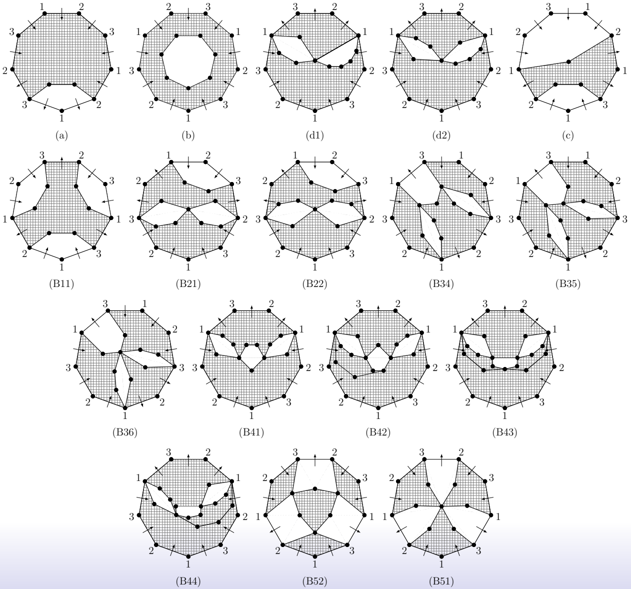

Photos of random people

Choi Ekstein ????? Holub Lidický

Proof Idea

Theorem (Tutte 1954)

A plane graph G has a 3-coloring if and only if its dual $G^\star$ has a nowhere-zero $\mathbb{Z}_3$-flow.

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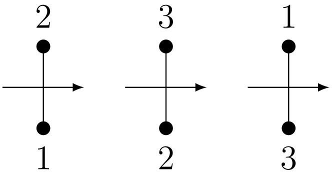

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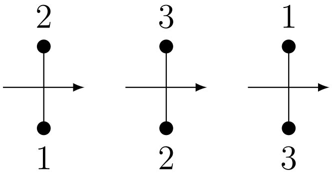
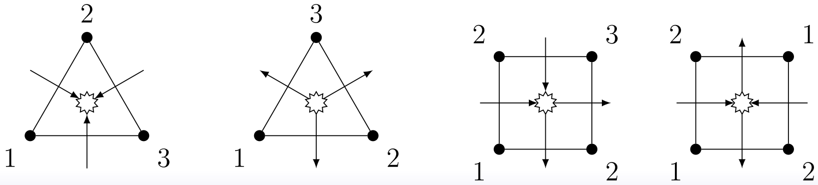


(In-edges - out-edges) of every face is a multiple of 3!



Theorem (Gimbel–Thomassen 1997, Aksenov–Borodin–Glebov 2003)

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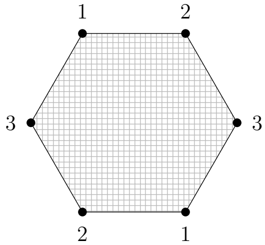

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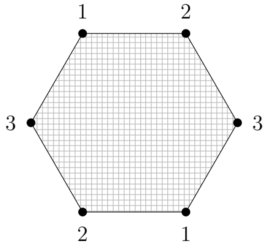

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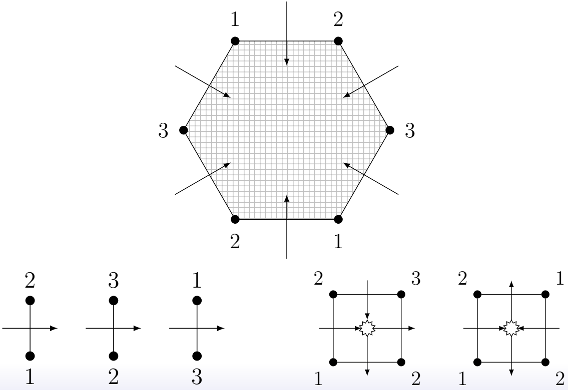

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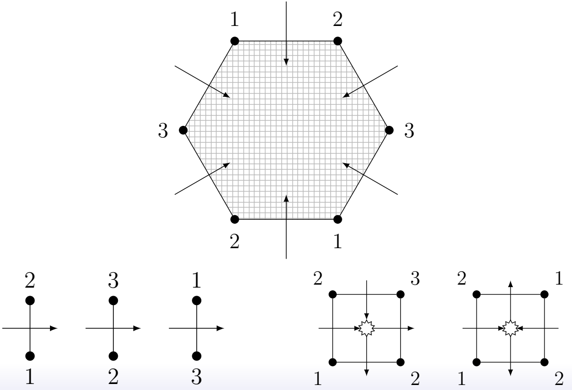

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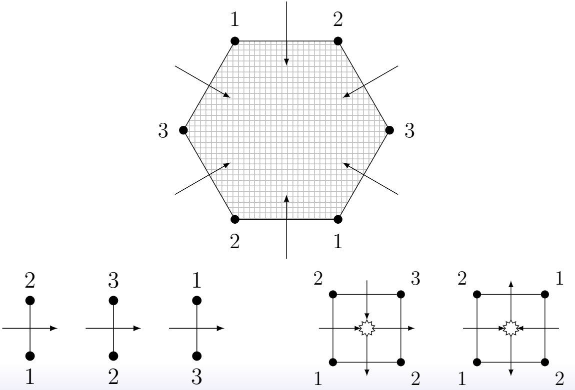

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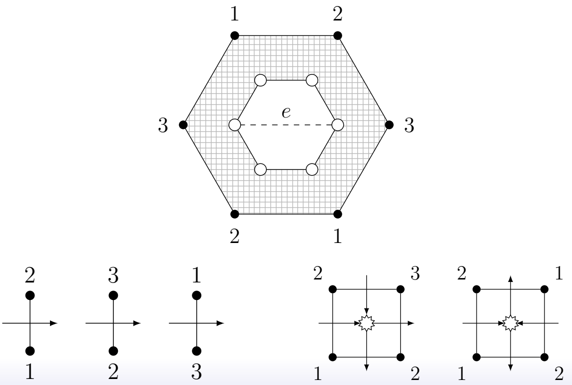

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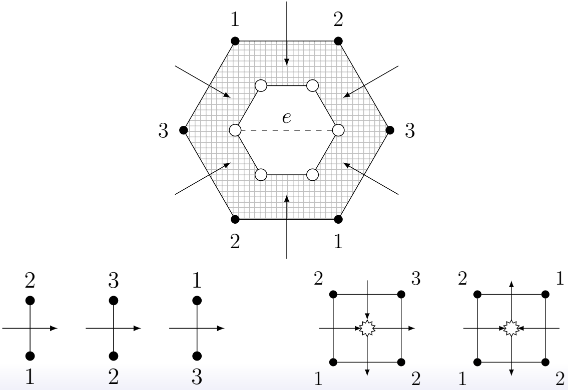

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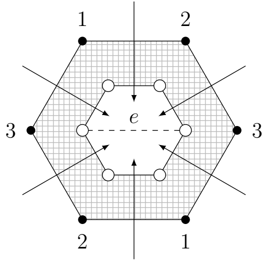

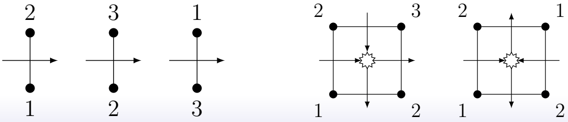

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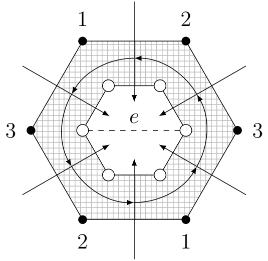

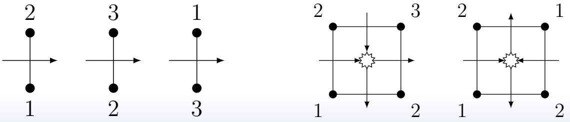

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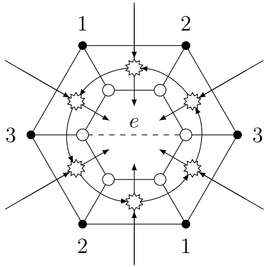

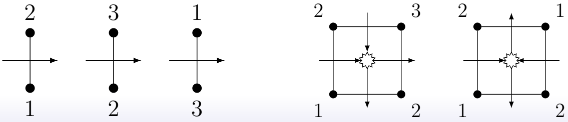

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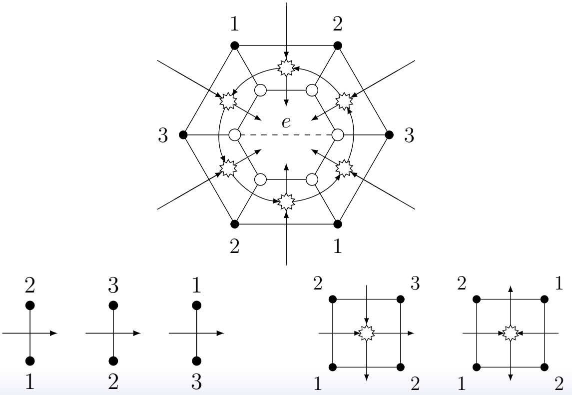

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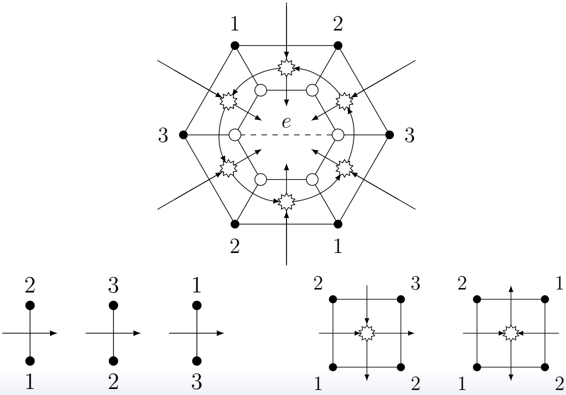
(⇒) ?



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(⇒) done!

Corollary (Dvořák–Král'–Thomas 2014+)

If G is a "nice" plane graph of girth 4 bounded by a cycle C and G is C-critical, then

$$
\begin{aligned}
|C| = 6 &: \quad \emptyset \\
|C| = 7 &: \quad \{5\} \\
|C| = 8 &: \quad \emptyset, \{6\}, \{5, 5\} \\
|C| = 9 &: \quad \{7\}, \{5, 6\}, \{5, 5, 5\}, \{5\}
\end{aligned}
$$

are the only possible multisets of faces of length at least 5.

Corollary (Dvořák–Kráľ–Thomas 2014+)

If G is a "nice" plane graph of girth 4 bounded by a cycle C of length 9 and is C-critical, then

$$\{7\}, \{5,6\}, \{5,5,5\}, \{5\}$$

are the only possible multisets of faces of length at least 5.

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Theorem (Choi–Ekstein–Holub–Lidický 2014+)

If G is a "nice" plane graph of girth 4 bounded by a 9-cycle C containing a 5-face and a 6-face, then G is C-critical if and only if G "looks like" a graph below.

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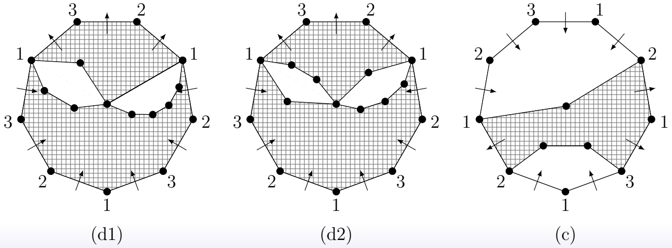

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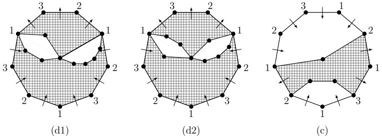

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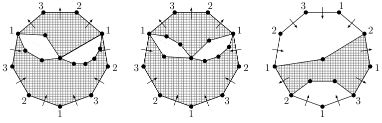

(d1) (d2) (c)

Proof: ($\Rightarrow$)

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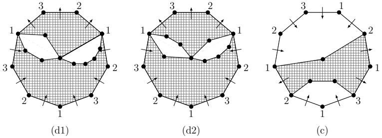
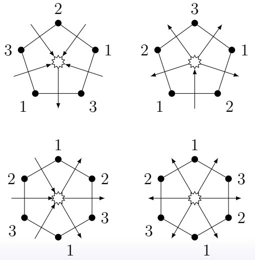


Proof: ($\Rightarrow$) "Tutte's Flow Theorem!"



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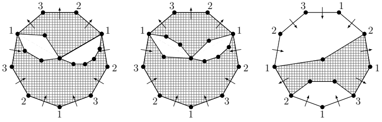

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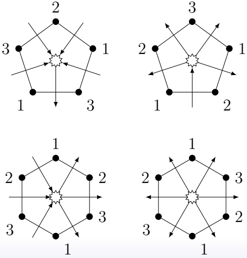

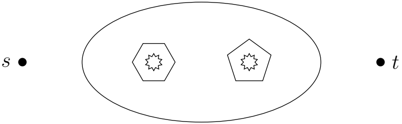

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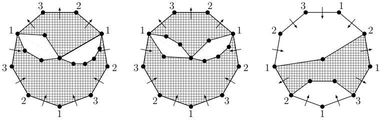

(d1) (d2) (c)

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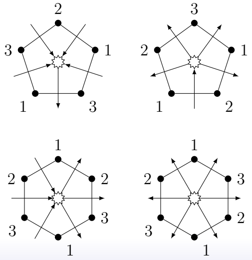

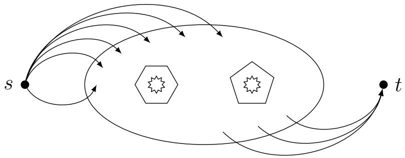

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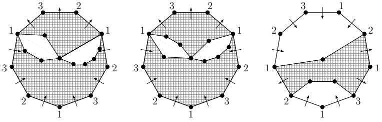

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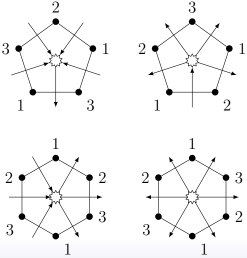

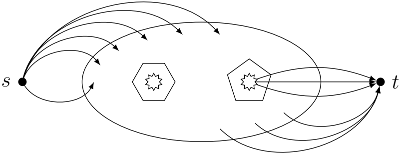

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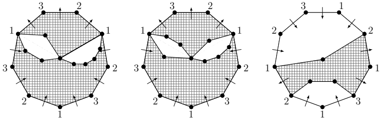

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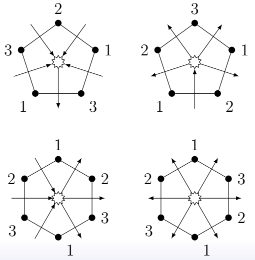

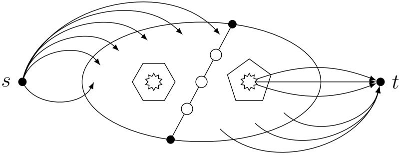

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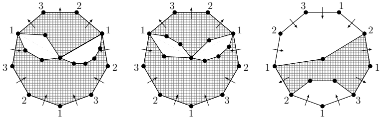

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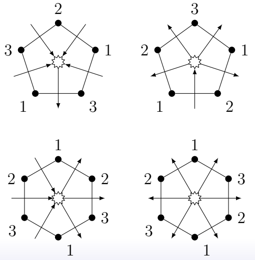

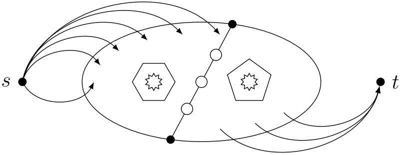

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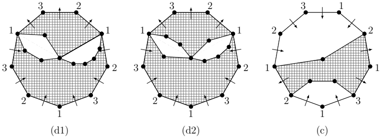

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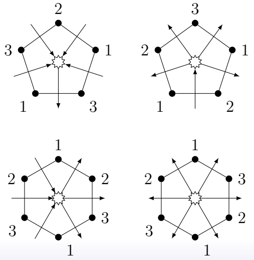

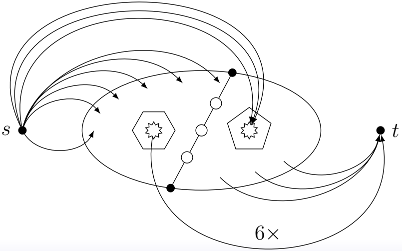

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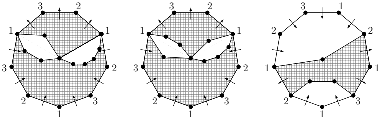

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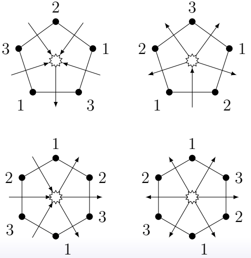

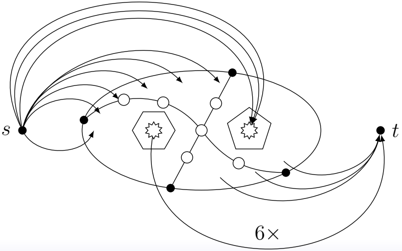

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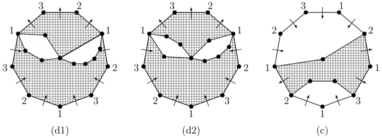

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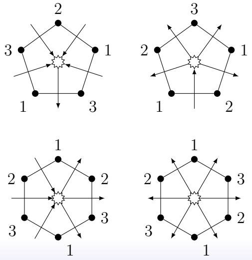

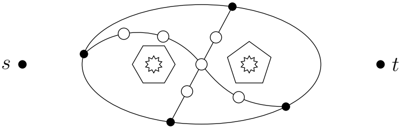

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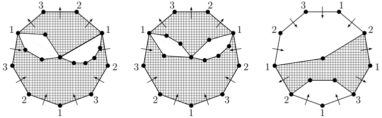

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Proof: $(\Leftarrow)$

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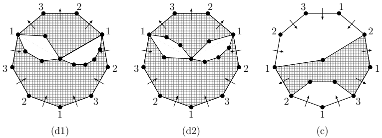
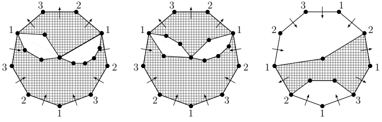


Proof: ($\Leftarrow$) Check each one!

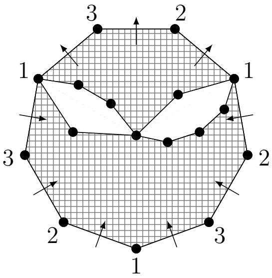
Theorem (Choi–Ekstein–Holub–Lidický 2014+)

If G is a "nice" plane graph of girth 4 bounded by a cycle C of length 9 containing a 5-face and a 6-face, then G is C-critical if and only if G "looks like" a graph below.

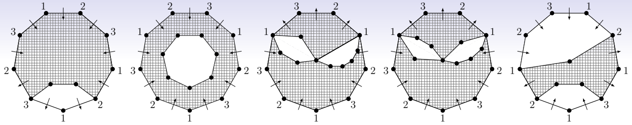


(d1) (d2) (c)

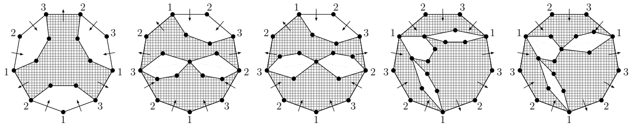
Proof: ($\Leftarrow$) Check each one!



(d2)

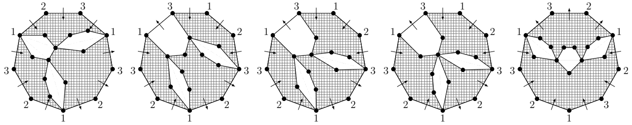


(a) (b) (d1) (d2) (c)

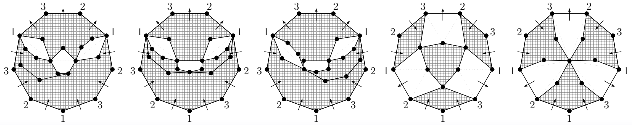


(B11) (B21) (B22) (B31) (B32)

Thank you for your attention!



(B33) (B34) (B35) (B36) (B41)



(B42) (B43) (B44) (B52) (B51)