Precoloring extension for planar graphs

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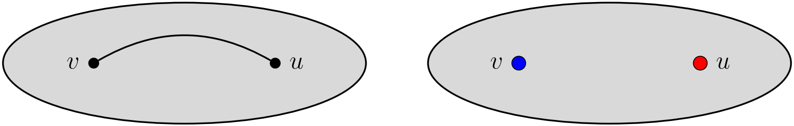
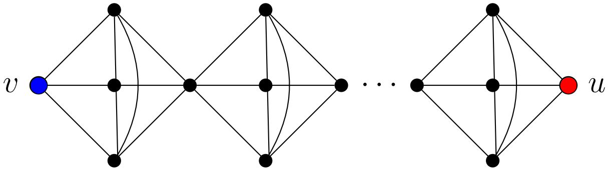
Theorem (The Four Color Theorem)

Every planar graph can be colored by 4 colors.

Are 4 colors enough for planar plus one edge?



Is it possible to precolor 2 vertices?

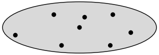


Theorem (Albertson)

Let G be a planar graph and $S \subset V(G)$. If $\forall u, v \in S$ $dist(u, v) \geq 4$ then any precoloring of S extends to a 5-coloring of G.

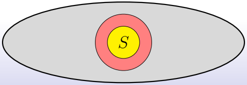
Theorem (Dvořák, L., Postle, Mohar)

Let G be a planar graph and $S \subset V(G)$. If $\forall u, v \in S$ $dist(u, v) \geq 50000$ then any precoloring of S extends to a 5-list-coloring of G.



Theorem (Postle, Thomas)

If G is planar, $S \subset V(G)$ and a precoloring of S extends to all vertices at distance $\Omega(\log|S|)$ from S, then it extends to a 5-coloring of G.



Theorem (Grötzsch)

Every triangle-free planar graph is 3-colorable.

Theorem (Aksenov; Jensen and Thomassen)

Precoloring of any two vertices in triangle-free planar graph extends to a 3-coloring.

Theorem (Aksenov)

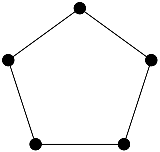
Every planar graph with at most 3 triangles is 3-colorable.

Theorem (Borodin, Dvořák, Kostochka, L., Yancey)

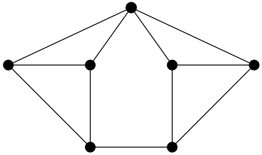
Precise description of 4-critical planar graphs with exactly 3 triangles.

Definition (k-critical graphs)

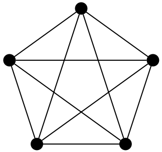
Graph G is a *k-critical graph* if G is not $(k-1)$-colorable but every $H \subset G$ is $(k-1)$-colorable.



3-critical



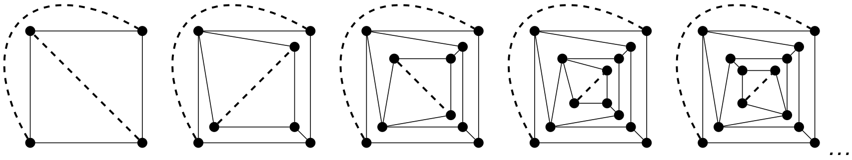
4-critical



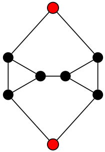
5-critical

Theorem (Borodin, Dvořák, Kostochka, L., Yancey)

All plane 4-critical graphs with 4 triangles and no 4-faces can be obtained from the Thomas-Walls sequence



by replacing dashed edges by edges or by Havel's quasiedge:



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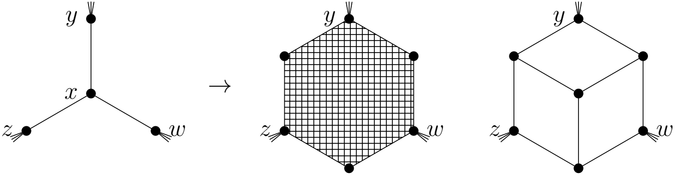
Theorem (Borodin, Dvořák, Kostochka, L., Yancey)

All plane 4-critical graphs with 4 triangles and no 4-faces $\mathcal{C}$ can be obtained from the Thomas-Walls sequence by replacing dashed edges by edges or by Havel's quasiedge.

Theorem (Borodin, Dvořák, Kostochka, L., Yancey)

Every 4-critical plane graph with 4-triangles can be obtained from $G \in \mathcal{C}$ by expanding some vertices of degree 3. (called patching)



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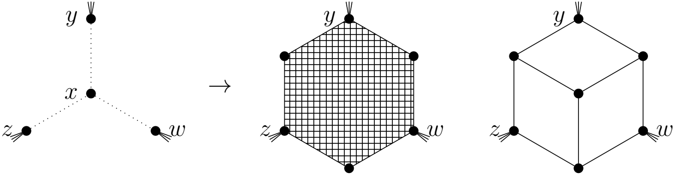

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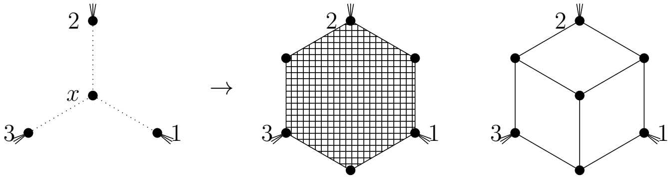

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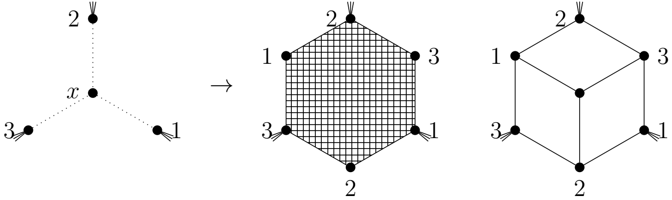
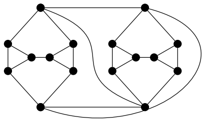


Conjecture (Grunbaum 63)

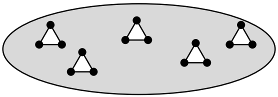
Planar graphs without intersecting triangles are 3-colorable.

Disproved by Havel



Conjecture (Havel)

If G is planar and mutual distance of triangles is $\geq O(1)$, then G is 3-colorable.



Proved by Dvořák, Král', Thomas.

Theorem (Dvořák, Kráľ, Thomas)

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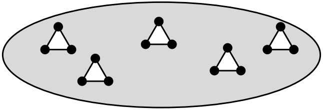
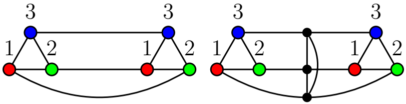
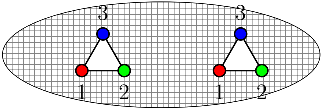


But triangles cannot be precolored.

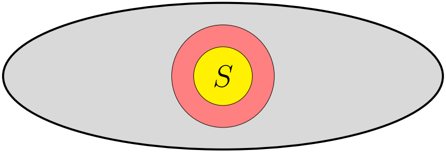




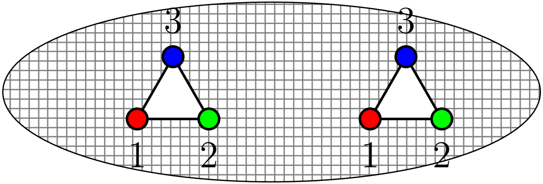
Winding number in quadrangulation.

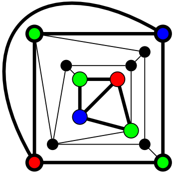
Theorem (Dvořák, Kráľ, Thomas)

If G is planar of girth 5, $S \subset V(G)$ and a precoloring of S extends to all vertices at distance $\Omega(|S|)$ from S, then it extends to a 3-coloring of G.



Girth 5 needed:

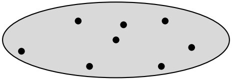




Postle: distance $100|S|$ is siffucient.

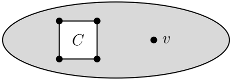
Conjecture (Dvořák, Kráľ, Thomas)

If G is planar of girth 4, $S \subset V(G)$ and $\forall u, v \in S$ $dist(u, v) \geq \Omega(1)$ then any precoloring of S extends to a 3-coloring of G.



Conjecture (Dvořák, Kráľ, Thomas)

If G is planar of girth 4, $S \subset V(G)$, S consists of a vertex v and 4-cycle C, and distance of v and C is $\geq \Omega(1)$ then any precoloring of S extends to a 3-coloring of G.



Theorem (Dvořák, Kráľ, Thomas)

Second conjecture implies the first one.

- We prove the second conjecture

Theorem (Dvořák, L.)

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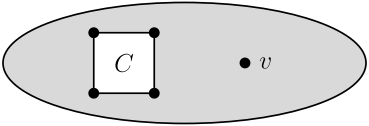
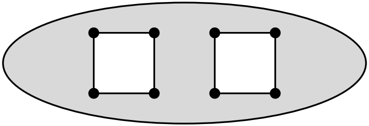


Characterize when a precoloring of two 4-cycles extend.

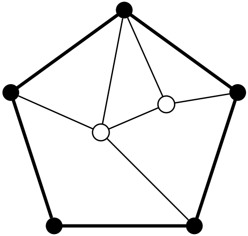


Definitions (S-critical graphs)

graph $G = (V, E)$, $S \subset G$

G is *S-critical graph* for 3-coloring if for every $S \subset H \subset G$ exists a 3-coloring φ of S such that

- φ extends to a 3-coloring of H
- φ does not extend to a 3-coloring of G.



Note that $\emptyset$-critical graph is 4-critical.

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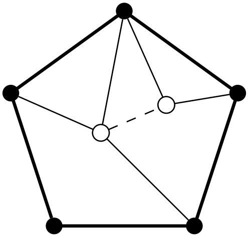

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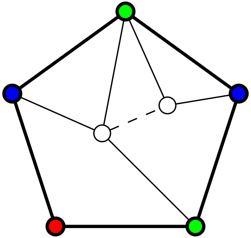

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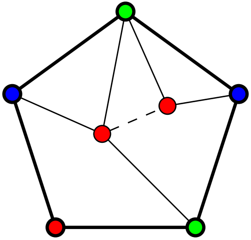

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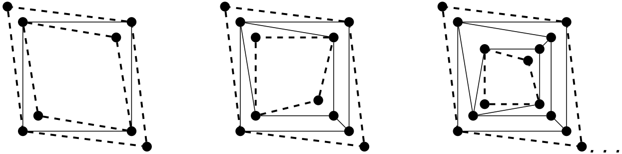
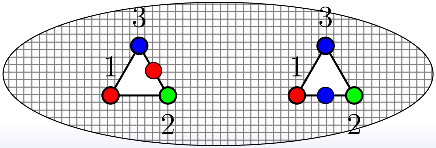
Theorem (Dvořák, L.)

Let G be planar graph with two 4-faces C_1 and C_2 in distance $\geq \Omega(1)$. If all triangles in G are disjoint, non-cotractible, and G is (C_1, C_2)-critical,

- *G is obtained from framed patched Thomas-Walls, or*

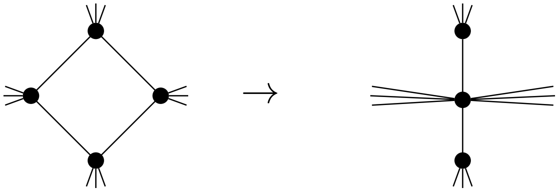


- *G is near $3, 3$-quadrangulation.*

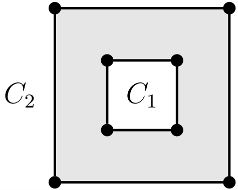


Proof sketch

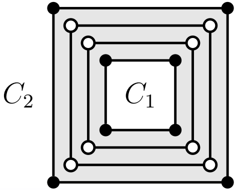
Main tool - collapse 4-faces



Few separating $\leq$ 4-cycles



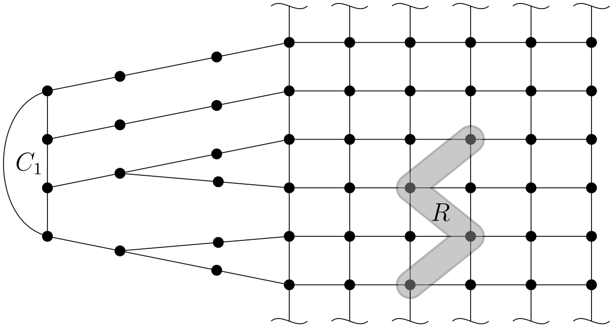
Many separating $\leq$ 4-cycles



Proof sketch - creating separating $\leq$ 4-cycles

Lemma

If C_1 and C_2 are in distance d, then it is possible to create separating 4-cycle and decrease d by one.

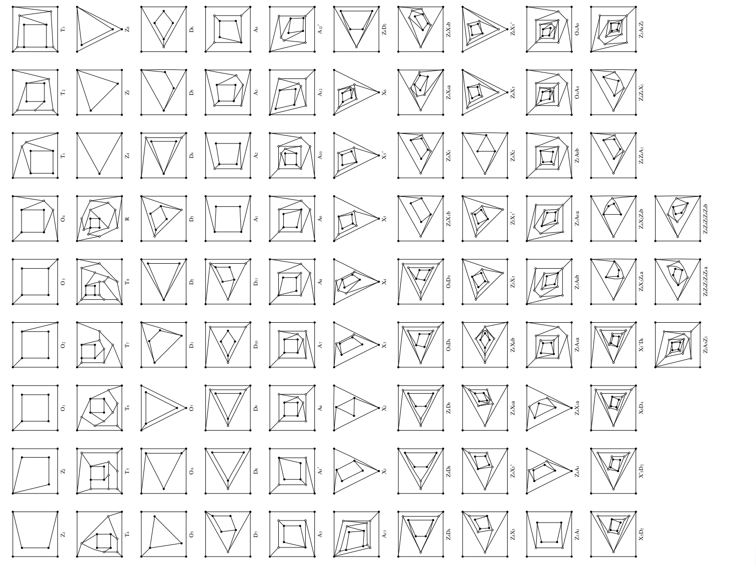


Proof sketch - many separating $\leq$ 4-cycles

- collapse 4-faces without destroying separating $\leq$ 4-cycles

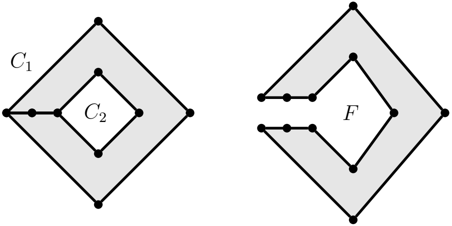
- describe *basic* graphs between separating $\leq$ 4-cycles
  - with 4-faces (21 graphs)
  - without 4-faces (94 graphs)
- gluing of at least $\geq$ 1056 (or $\geq$ 40 with computer) basic graphs
  - extends any precoloring of outer cycles, or
  - is Thomas-Walls, or
  - is almost $3, 3$-quadrangulation.

Exeptions without 4-faces



Lemma (Dvořák, L.)

Let G be a plane graph and C_1 and C_2 faces of G. If G is $(C_1 \cup C_2)$-critical and C_1 and C_2 are the only ≤ 4 cycles and the distance between C_1 and C_2 is at most 4 then G is one of 22 graphs.

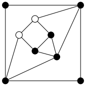


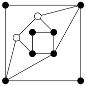
Results in a planar F-critical graph of girth 5 (outer face F).

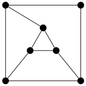
Theorem (Dvořák, L.)

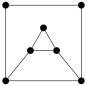
There are 7969 F-critical planar graphs of girth 5 with outer face F of size 16.

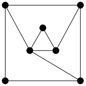
Exeptions with 4-faces

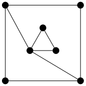

EV_1


EV_2


T_1


T'_1


T_2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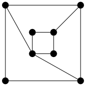
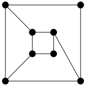
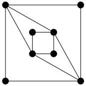
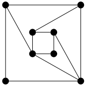
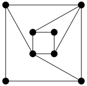
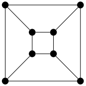

S_1

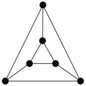

S_2

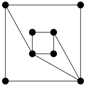

Q_1

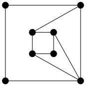

Q_2

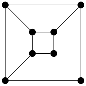

Q_3

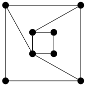

Q_4

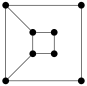

Q_5

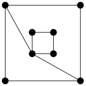

X_1

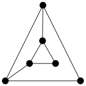

X_2

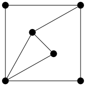

X_3

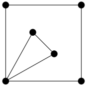

X_4


X_5

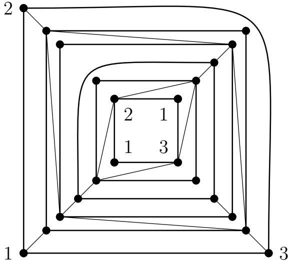

X_6


X_7

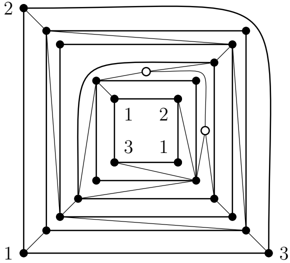

J_1


I_4

Chains of small graphs



$T_1\ Q_1\ T_1\ O_6\ Q_1$

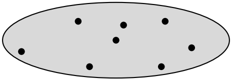


$T_1\ Q_1\ T_1\ EV_1\ S_1$

More consequences

Corollary (Dvořák, L.)

If G is an n-vertex triangle-free planar graphs with maximum degree Δ then G has at least $\left(3^{1/\Delta^D}\right)^n$ distinct colorings, where D is constant.

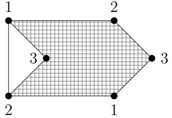


Pick n/Δ^D vertices S of G in mutual distance $\geq D$.
All $3^{|S|}$ precolorings of S extend to different 3-colorings of G.

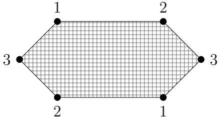
More consequences

Let G be a graph with a face C and t triangles. (3 coloring)

- if $|C| = 4$ and $t \leq 1$, any precoloring of C extends
- if $|C| = 5$ and $t \leq 1$, the only C-critical graph is



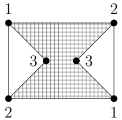
- if $|C| = 6$ and $t = 0$, the only C-critical graph is



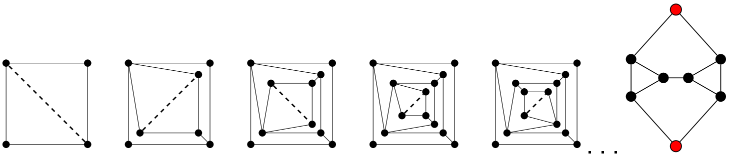
- if $|C| \leq 9$ and $t = 0$, C-critical graphs known

Theorem (Dvořák, L.)

Let G be a graph with a 4-face C and 2 triangles. If G is C-critical the G is



or framed patched Thomas-Walls, where the dashed edge is a normal edge or Havel's quaziedge.



(C is the outer face, vertices of degree 2 have different colors)

Thank you for your attention!