Triangles and 3-coloring of planar graphs

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Iowa State University

February 4, 2015

Outline

- what is graph coloring?
- Grötzsch's theorem: triangle-free planar graphs are 3-colorable
- extensions of GT preserving triangle-free
- extensions of GT allowing (few) triangles

Old theorems with new simple(r) proofs.
(some new theorems too)

Inspiration - coloring a political map

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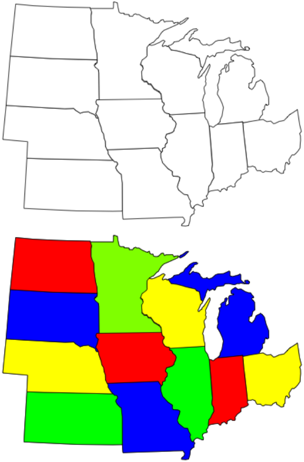

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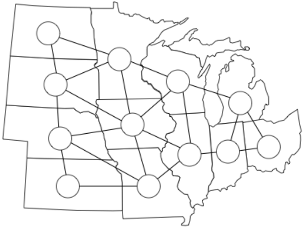

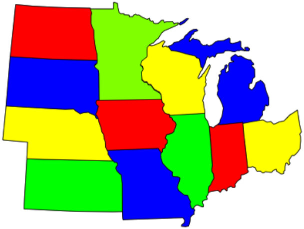

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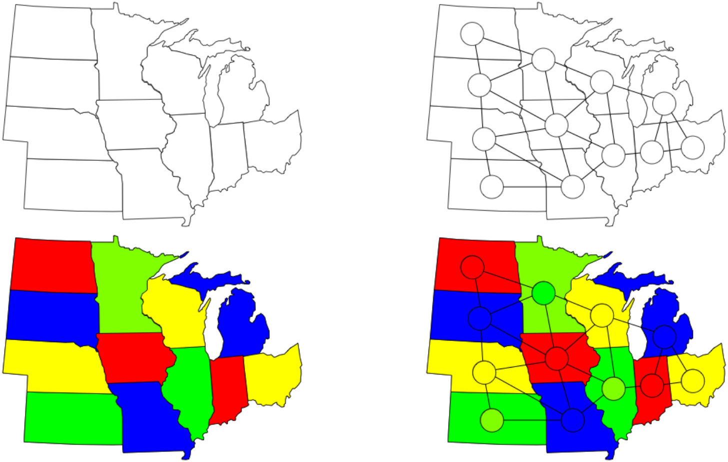

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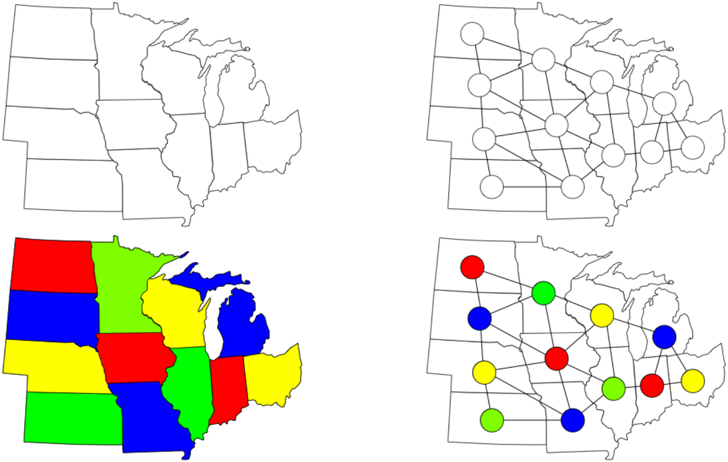

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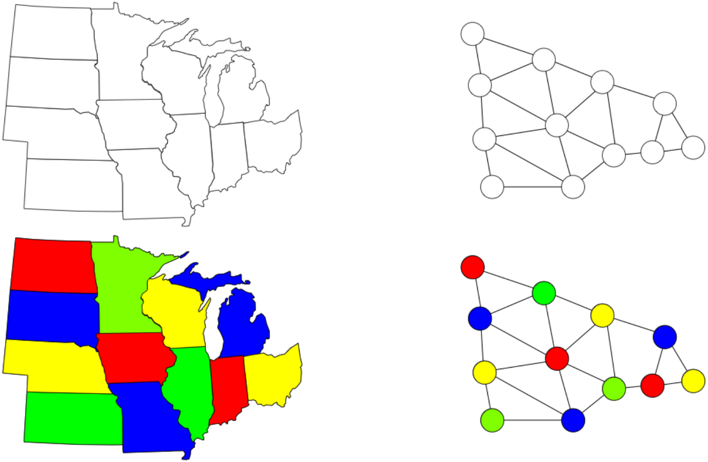
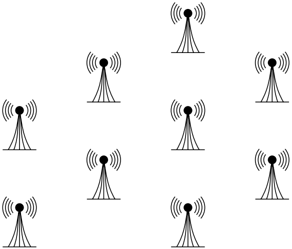


Task: Color vertices of a graph such that adjacent vertices have distinct colors.

Applications of graph coloring

Cellphone towers



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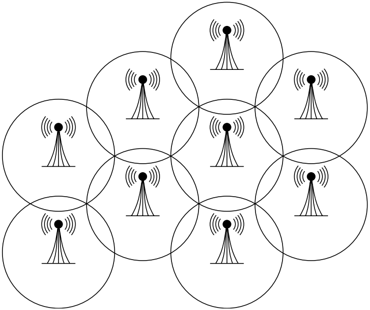

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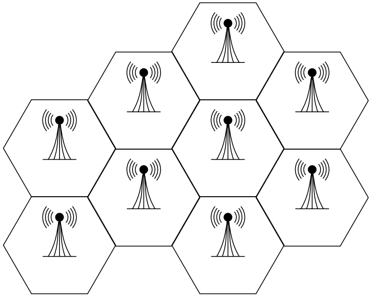

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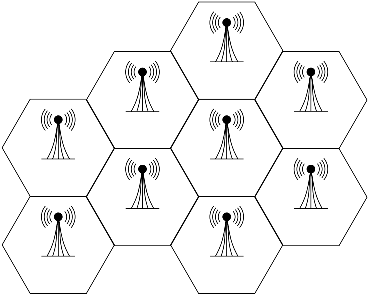

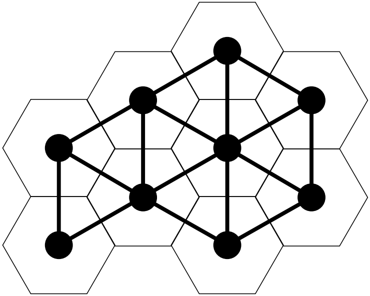

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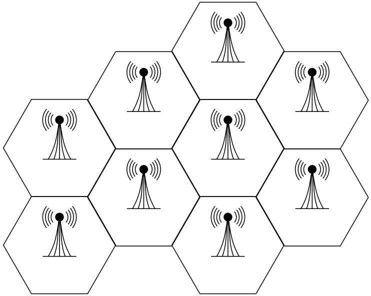

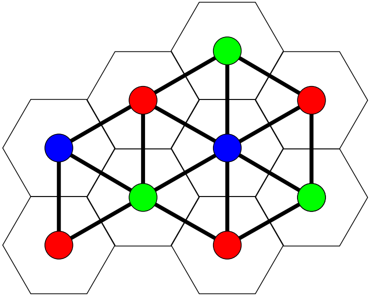

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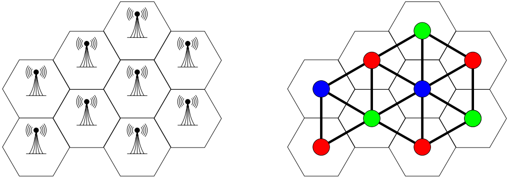
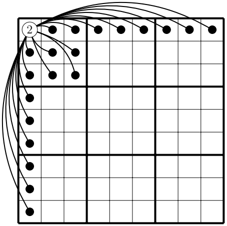


Scheduling, register allocation (code generating), DNA sequencing, . . . , Sudoku

2	7	6					1	
			4			8		
			1		5		2	
		1	6				3	9
9	4				3	1		
	9		3		6			
		2			4			
	8					3	9	2

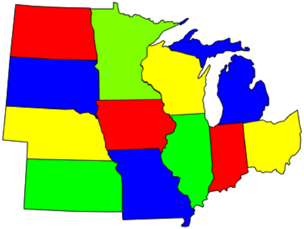


Sudoku: Extending a partial 9-coloring (precoloring).

The Four Color Theorem

Conjectured (Guthrie 1852)

Every planar graph can be (properly) colored using 4 colors.



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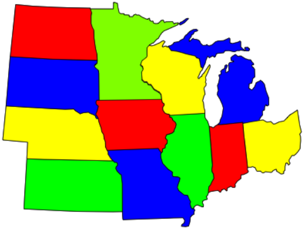

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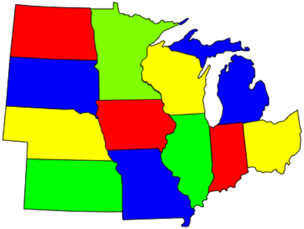

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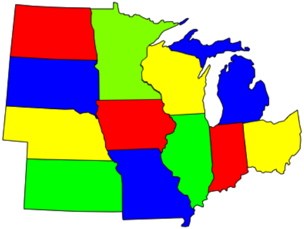

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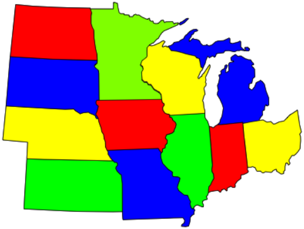

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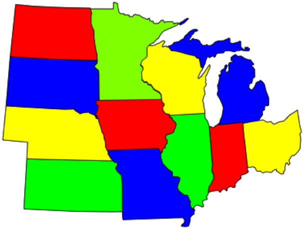

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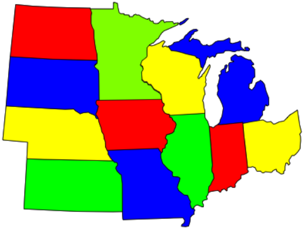

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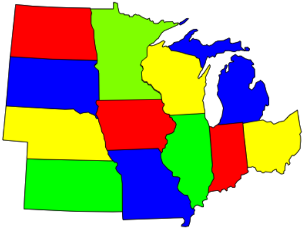

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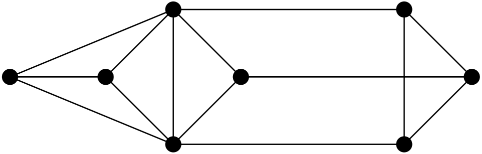
- proved by discharging

Graph coloring

- describing classes of graphs that are k-colorable
- describing efficiently k-colorable classes of graphs
- algorithms for coloring
- variants for applications

Definitions

A graph is *plane* if it is drawn without crossing edges.



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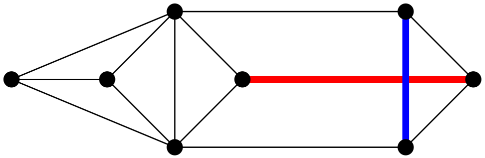

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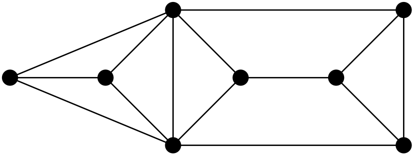

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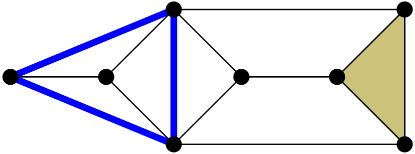

triangle = 3-cycle; 3-face

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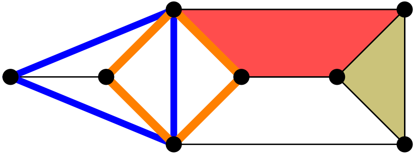

triangle = 3-cycle; 3-face; 4-cycle; 4-face

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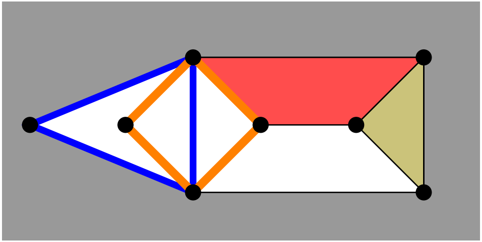
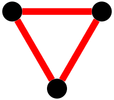


triangle = 3-cycle; 3-face; 4-cycle; 4-face; 5-face

Inspiration

Theorem (Grötzsch 1959)

Every planar triangle-free graph is 3-colorable.



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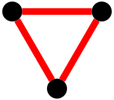


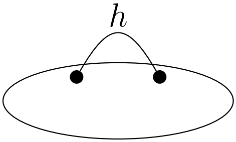
Generalizations:

- addition of an edge or a vertex
- precoloring subgraphs
- allowing some triangles

Adding a vertex or an edge

Theorem (Aksenov '77; Jensen, Thomassen '00)

If H can be obtained from a triangle-free planar graph by adding an edge h, then H is 3-colorable.



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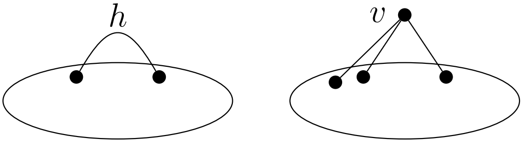

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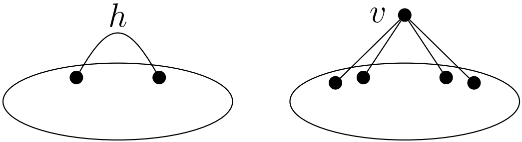

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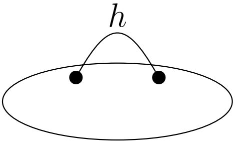

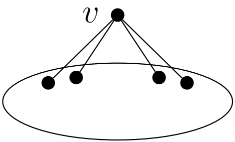

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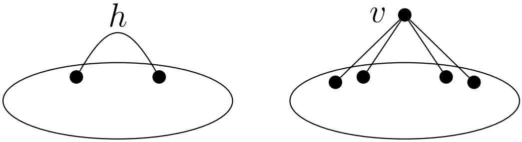
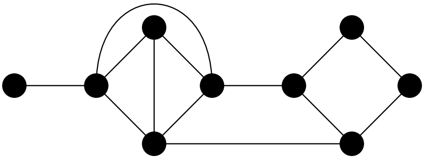


Both proofs similar.
Both theorems are tight.

Definition of 4-critical graph

Problem: How to efficiently describe graphs that are not 3-colorable?



A graph G is a *4-critical graph* if G is not 3-colorable but every $H \subset G$ is 3-colorable.

Useful as a minimal counterexample.

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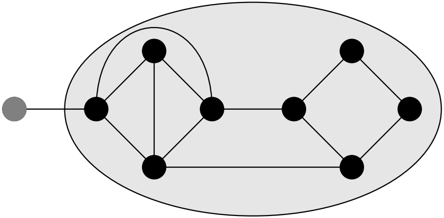

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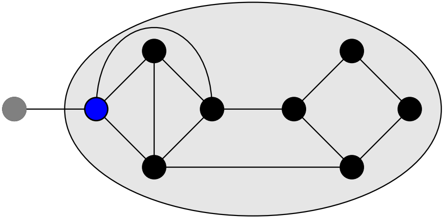

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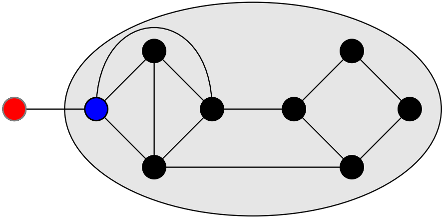

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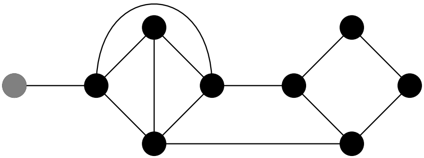

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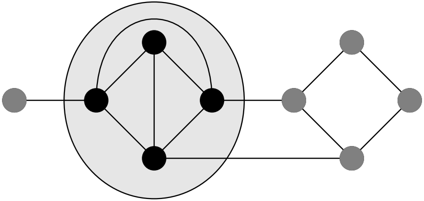

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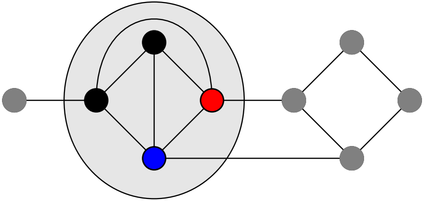

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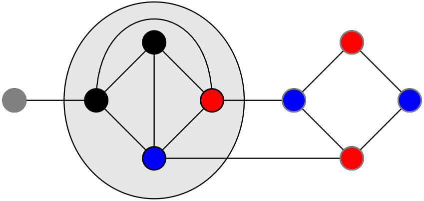

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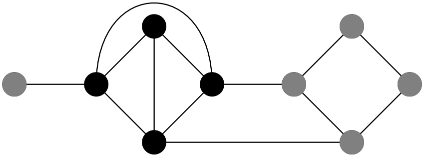

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Main tool

Theorem (Kostochka and Yancey '12+)

If G is a 4-critical graph, then

$$|E(G)| \geq \frac{5|V(G)| - 2}{3}.$$

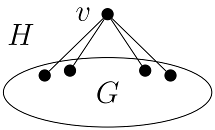
We write as $3|E(G)| \geq 5|V(G)| - 2$.

4*-critical graphs must have "many" edges*
G does not have to be planar

$V(G)$ is the vertex set of G and $E(G)$ is the edge set of G

Planar triangle-free graph and a 4-vertex

H is 4-critical, minimal counterexample
G plane, triangle-free, $G = H - v$



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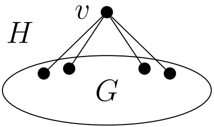
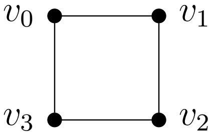


Case 1: G contains a 4-face (use minimality to 3-color H)



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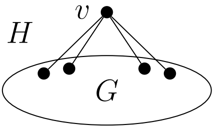

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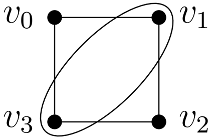
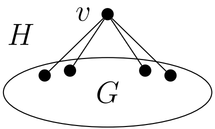


Planar triangle-free graph and a 4-vertex

H is 4-critical, minimal counterexample
G plane, triangle-free, $G = H - v$



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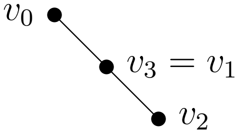

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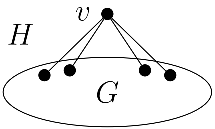

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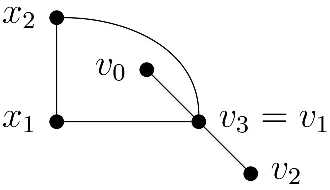

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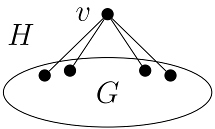
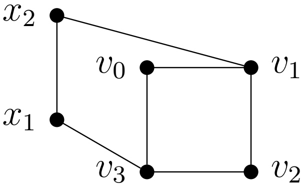


Case 1: G contains a 4-face (use minimality to 3-color H)



Planar triangle-free graph and a 4-vertex

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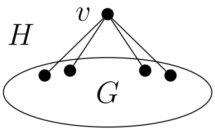

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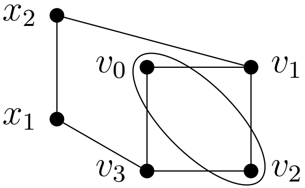

Planar triangle-free graph and a 4-vertex

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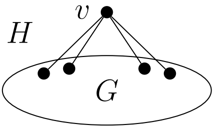

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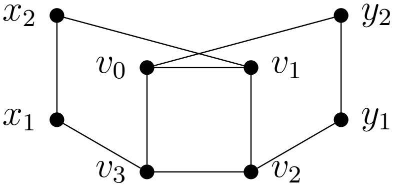

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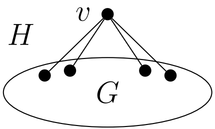

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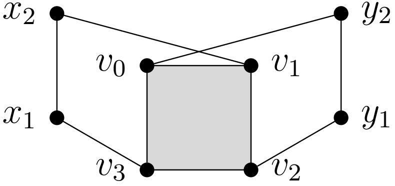

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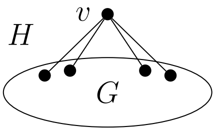

Case 1: G contains a 4-face
Case 2: G contains no 4-faces

$|E(G)| = e, |V(G)| = v, |F(G)| = f$
$F(G)$ is the set of faces of G

- $v - 2 + f = e$ by Euler's formula

Planar triangle-free graph and a 4-vertex

H is 4-critical, minimal counterexample
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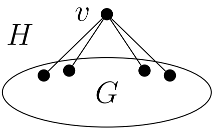

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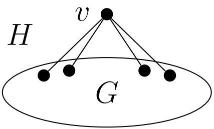

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- $5v - 10 + 5f = 5e$

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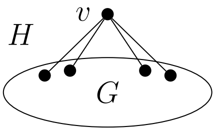

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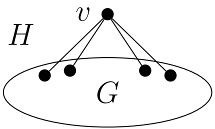

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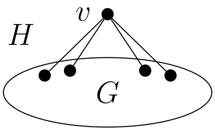
G plane, triangle-free, $G = H - v$



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- $3(e + 4) \geq 5(v + 1) - 2$ (H is 4-critical graph)

Planar triangle-free graph and a 4-vertex

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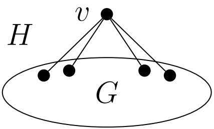

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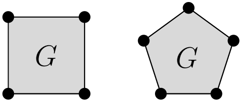
- $3(e + 4) \geq 5(v + 1) - 2$ (H is 4-critical graph)
- $5v - 10 \geq 3e \geq 5v - 9$, contradiction $\square$

Precoloring

Theorem (Grötzsch '59)

Every precoloring of a face of length at most 5 *in any triangle-free plane graph* G *can be extended to a (proper)* 3*-coloring of* G*.*



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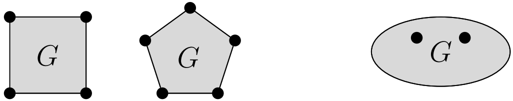
Theorem (Aksenov, Borodin, Glebov '02)

Every precoloring of two non-adjacent vertices in any triangle-free planar graph G *can be extended to a (proper)* 3*-coloring of* G*.*



Precoloring

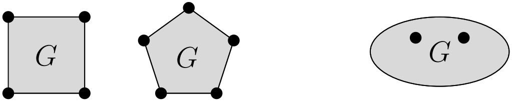
Theorem (Grötzsch '59; BKLY '14)

Every precoloring of a face of length at most 5 in any triangle-free plane graph G can be extended to a (proper) 3-coloring of G.

Theorem (Aksenov, Borodin, Glebov '02; BKLY '14)

Every precoloring of two non-adjacent vertices in any triangle-free planar graph G can be extended to a (proper) 3-coloring of G.

(Proof similar to the previous one.)



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Theorem (Grötzsch '59; BKLY '14)

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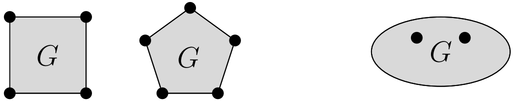
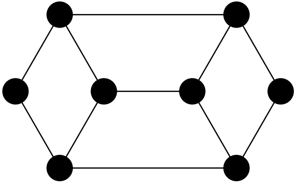


Both theorems are tight.

Tightness for precoloring a 6-face



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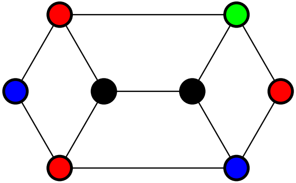

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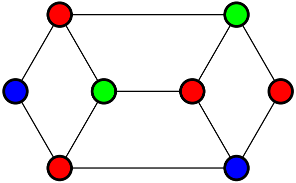

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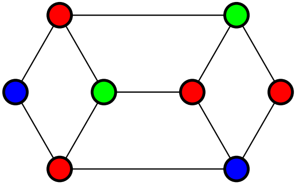

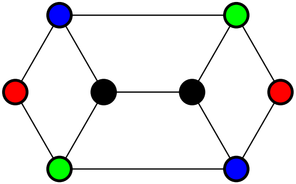

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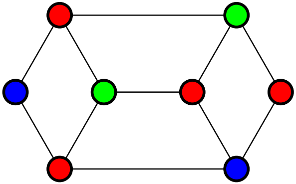

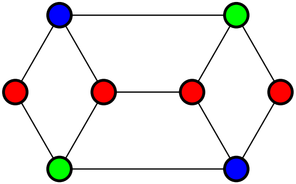

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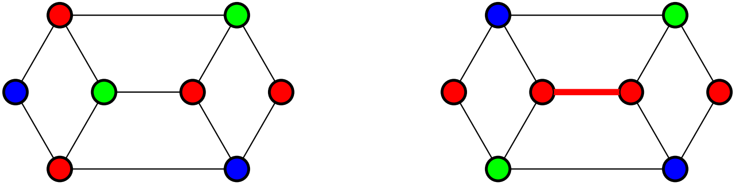
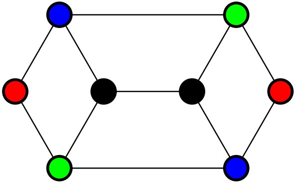


Not every precoloring of a 6-face extends to a 3-coloring.

Precoloring larger face

Description of “critical” graphs with precolored

- 6-face by Gimbel, Thomassen ’97;
  Aksenov, Borodin, Glebov ’03



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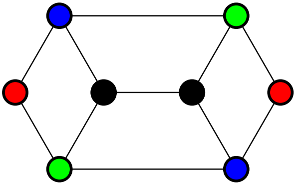

- 7-face by Aksenov, Borodin, Glebov ’04 (discharging);
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Precoloring larger face

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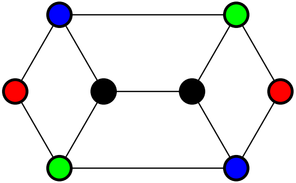

- 7-face by Aksenov, Borodin, Glebov ’04 (discharging);
  Dvořák, **L.** ’14 (network flows)
- 8-face by Dvořák, **L.** ’14

Precoloring larger face

Description of “critical” graphs with precolored

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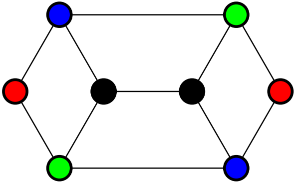

- 7-face by Aksenov, Borodin, Glebov ’04 (discharging);
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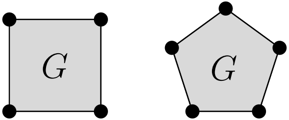
- 8-face by Dvořák, **L.** ’14
- 9-face by Choi, Ekstein, Holub, **L.** ’15+

New proof

Theorem (Grötzsch '59, BKLY '14)

Every precoloring of a face of length at most 5 in any triangle-free plane graph G can be extended to a (proper) 3-coloring of G.

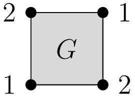


Our proof is significantly easier.

Proof

If G is a triangle-free planar graph and F is a precolored 4-face or 5-face, then the precoloring of F extends.

Case 1: F is a 4-face

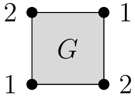


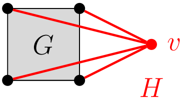
Case 2: F is a 5-face

Proof

If G is a triangle-free planar graph and F is a precolored 4-face or 5-face, then the precoloring of F extends.

Case 1: F is a 4-face H is 3-colorable





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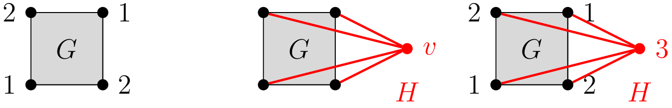

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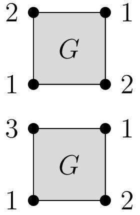

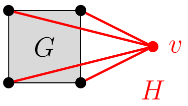

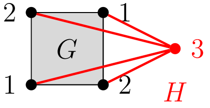

Case 2: *F* is a 5-face

Proof

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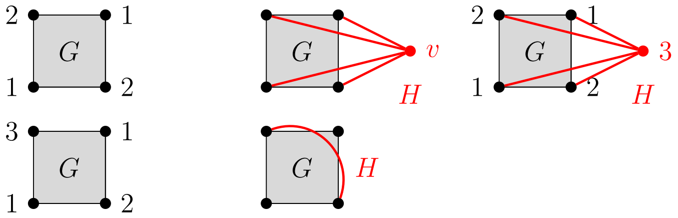
Case 1: F is a 4-face H is 3-colorable



Case 2: F is a 5-face

Proof

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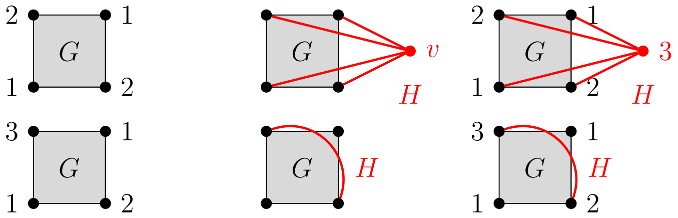
Case 1: F is a 4-face H is 3-colorable



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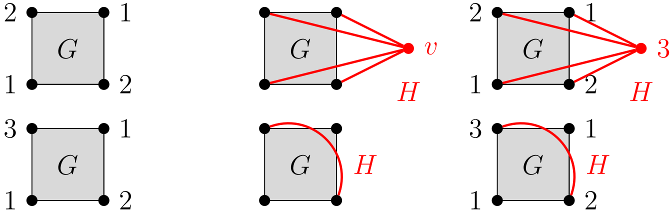

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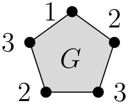

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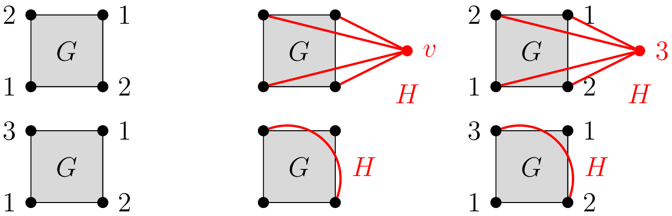
If G is a triangle-free planar graph and F is a precolored 4-face or 5-face, then the precoloring of F extends.

Case 1: F is a 4-face H is 3-colorable



Case 2: F is a 5-face

Proof

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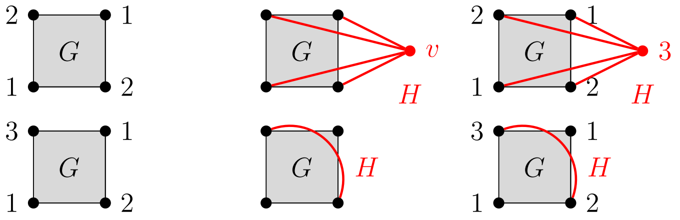

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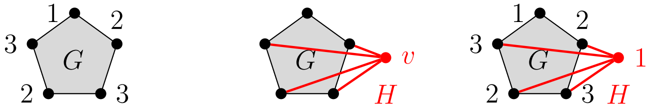

□

Allowing some triangles

Theorem (Grötzsch ’59)

Every planar triangle-free graph is 3-colorable.

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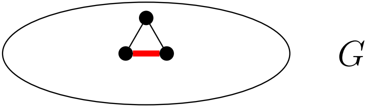
One triangle is easy!



Removing one edge of the triangle results in triangle-free G.

Allowing some triangles

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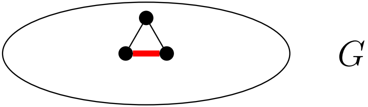
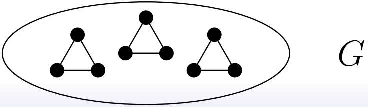


Removing one edge of the triangle results in triangle-free *G*.

Theorem (Grünbaum '63; Aksenov '74; Borodin '97; BKLY '14)

Every planar graph containing at most three triangles is 3-colorable.



Theorem (Grünbaum '63; Aksenov '74; Borodin '97; BKLY '14)

Every planar graph containing at most three triangles is 3-colorable.

Proof

- G is 4-critical (minimal counterexample)
- Reductions:
  - every 3-cycle is a face
  - every 4-cycle is a face or has a triangle inside and outside
  - every 5-cycle is a face or has a triangle inside and outside

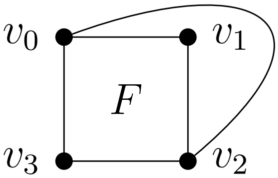
Case 1: G has no 4-faces

Case 2: G has a 4-face with a triangle (no identification)

Case 3: G has a 4-face where identification is possible

Three triangles - Proof sketch

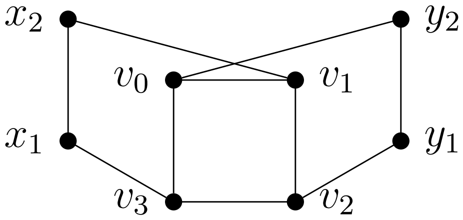
Case 2: G has a 4-face F with a triangle (no identification)



Both v_0, v_1, v_2 and v_0, v_2, v_3 are faces. G has 4 vertices!

Three triangles - Proof sketch

Case 3: G has a 4-face where identification is possible



Since G is plane, some of these vertices are the same.

Three triangles - Proof sketch

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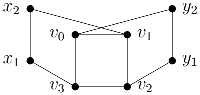
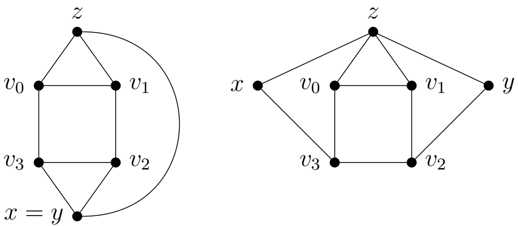


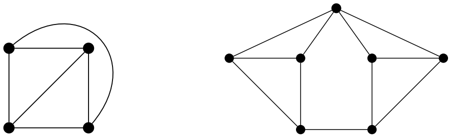
Since G is plane, some of these vertices are the same.
Only two cases left ...



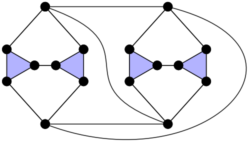
□

Problem (Erdős ’90)

Describe 4*-critical planar graphs containing* 4 *triangles.*



Havel ’69; Aksenov ’70s



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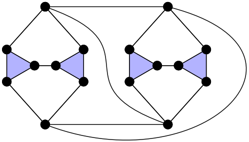
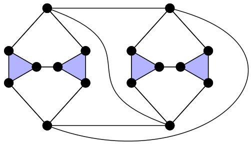
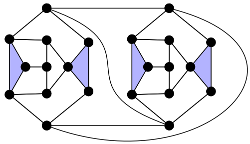


Problem (Sachs ’72)

Can the triangles be partitioned into two pairs so that in each pair the distance between the triangles is less than two?

Havel '69; Aksenov '70s; Aksenov, Melnikov '78,'80





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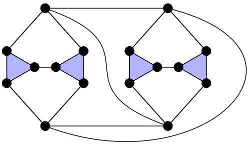

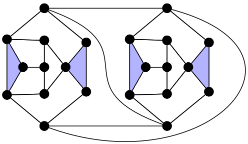

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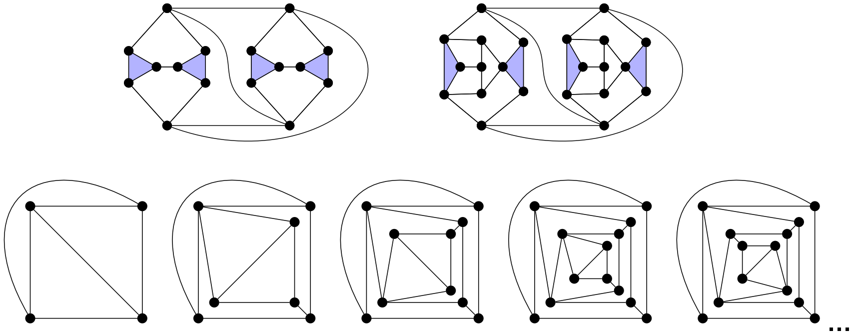

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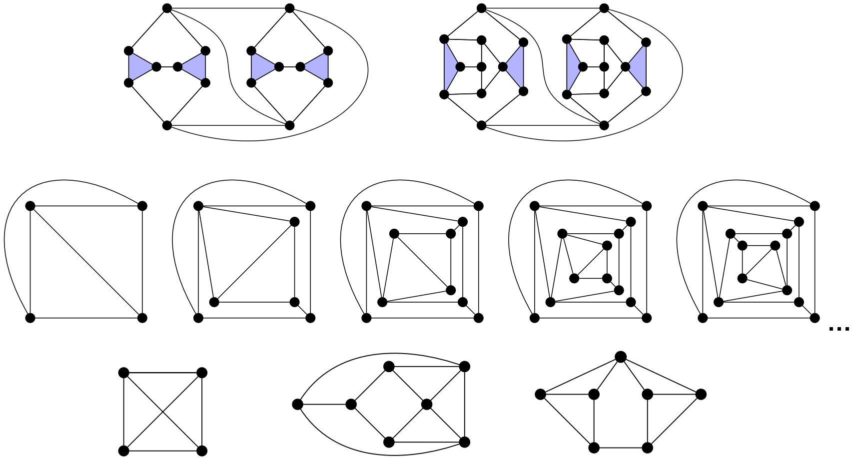

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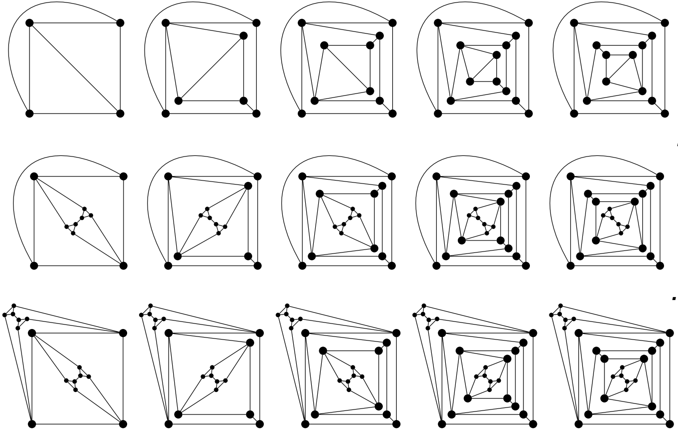
Theorem (Borodin, Dvořák, Kostochka, **L.**, Yancey ’14)

If G is 4-critical plane graph with 4 triangles and no 4-faces then it is one of



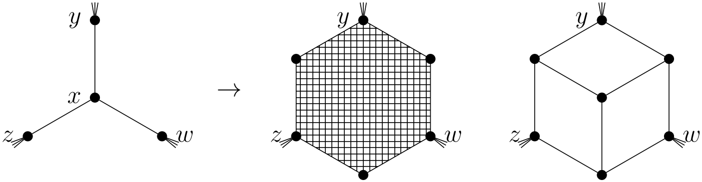
...

...

...

Theorem (Borodin, Dvořák, Kostochka, **L.**, Yancey '14)

Every 4-critical plane graph with 4 triangles can be obtained from a 4-critical plane graph G' with 4 triangles and no 4-faces by expanding some vertices of degree 3.



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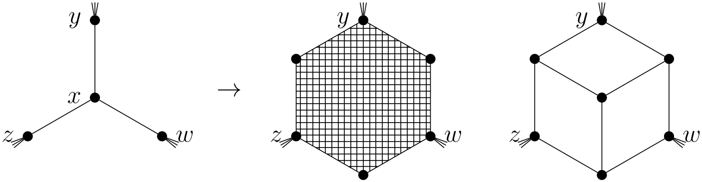

Corollary

Triangles can be partitioned into two pairs so that in each pair the distance between the triangles is ~~less than~~ at most two.

Grötzsch's theorem on surfaces

- triangle-free is not enough for 3-coloring on surfaces
- finitely (depends on genus) many 4-critical graphs if triangle-free and 4-cycle-free
- no contractible triangles and 4-cycles is enough for projective plane and torus

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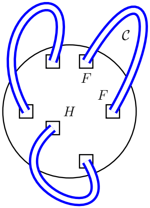
Theorem (Dvořák, **L.** '14)

Every 4-critical graph without contractible triangles and 4-cycles embedded in a surface of genus g looks like



where $|V(H)| = O(g)$, F are 4-cycles and $\mathcal{C}$ are from Thomas-Walls. (By discharging, computer assisted.)

Thank you for your attention!