3-COLORING TRIANGLE-FREE PLANAR GRAPHS

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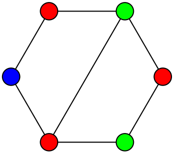
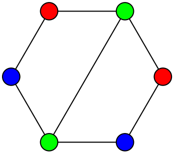
Sep 6, 2016

Coloring of a graph

Definition

A *(proper) coloring* of a graph G is a mapping $\varphi : V(G) \to C$ such that for every $uv \in E(G) : \varphi(u) \neq \varphi(v)$.

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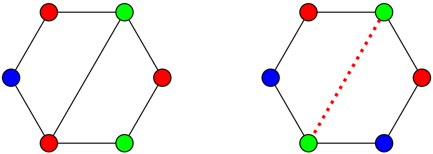

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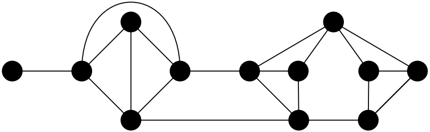
Definition of 4-critical graph

Problem: How do we efficiently describe graphs that are not 3-colorable?
What are obstacles for 3-coloring?



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Useful as a minimal counterexample.

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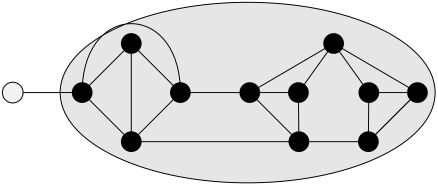

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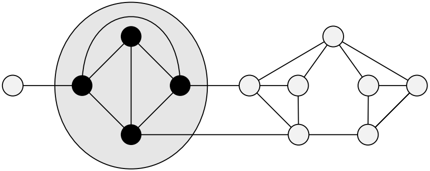

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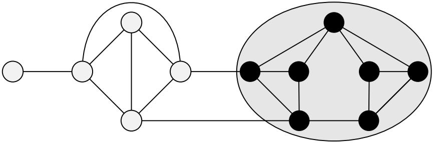

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Grötzsch's Theorem

Theorem (Appel, Haken '77)

Every planar graph is 4-colorable.

Theorem (Grötzsch '59)

Every triangle-free planar graph is 3-colorable.

Outline

- Proof of Grötzsch's Theorem
- Easy improvements
- Precolored faces
- Few triangles
- Precolored vertices

Proof of Grötzsch's Theorem - main tool

Theorem (Kostochka and Yancey '14)

If G is a 4-critical graph, then

$$|E(G)| \geq \frac{5|V(G)| - 2}{3}.$$

We write this as $3|E(G)| \geq 5|V(G)| - 2$.

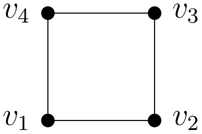
4-*critical graphs must have "many" edges*

G does not have to be planar

Proof of Grötzsch's Theorem (by K-Y)

G a minimal counterexample - plane, triangle-free, 4-critical.

Case 1: G contains a 4-face (try to 3-color G)



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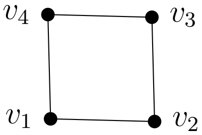

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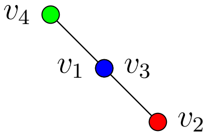

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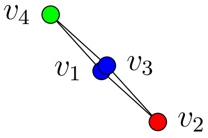

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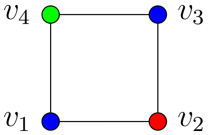

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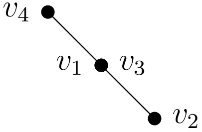

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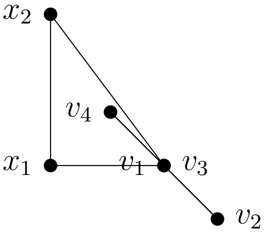

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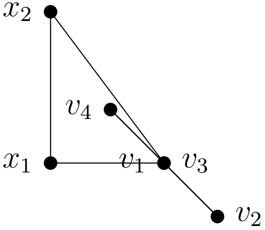

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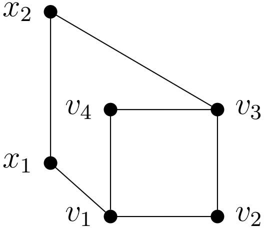

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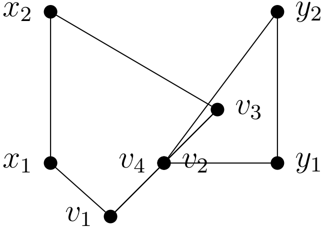

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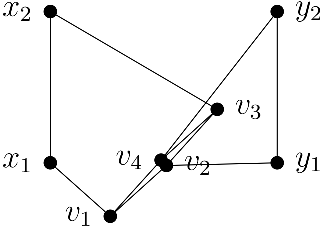

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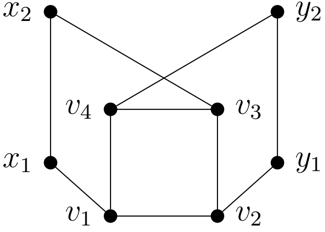

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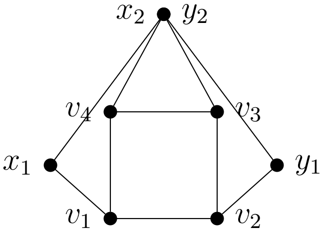


Case 2: G contains no 4-faces

$|E(G)| = e, |V(G)| = v, |F(G)| = f$.

- $v + f = e - 2$ by Euler's formula
- $5v - 10 \geq 3e$ (no 3-,4-faces)
- $3e \geq 5v - 2$ (every 4-critical graph)

Theorem (Grötzsch '59)

Every planar triangle-free graph is 3-colorable.

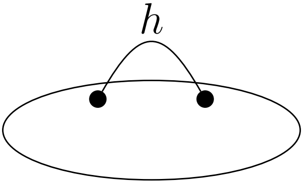
The last inequalities have a gap:

- $5v - 10 \geq 3e$ (no 3-,4-faces)
- $3e \geq 5v - 2$ (every 4-critical graph)

Plus extra edge

Theorem (Aksenov '77; Jensen, Thomassen '00)

If H can be obtained from a triangle-free planar graph by adding an edge h, then H is 3-colorable.



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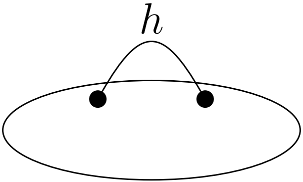
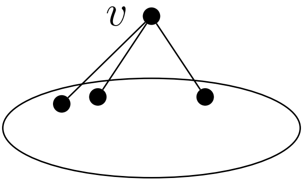


Simple proof (Borodin, Kostochka, L., Yancey 2014).
Tight K_4

Theorem (Jensen, Thomassen '00)

If H can be obtained from a triangle-free planar graph by adding a vertex v of degree 3, then H is 3-colorable.



Plus extra vertex

Theorem (Borodin, Kostochka, **L.**, Yancey '14)

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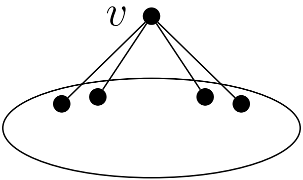
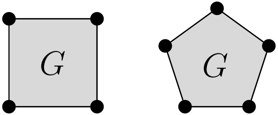


(and the proof is simple, tight by 5-wheel)

Theorem (Grötzsch '59)

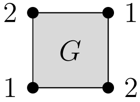
Let G be a triangle-free plane graph and F be the outer face of G of length at most 5. Then each 3-coloring of F can be extended to a 3-coloring of G.



Simple proof

If G is a triangle-free plane graph, F is a precolored external 4-face or 5-face, then the precoloring of F extends.

Case 1: F is a 4-face



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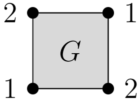

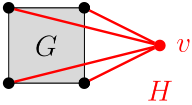

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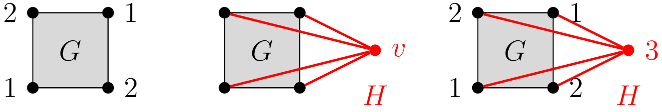

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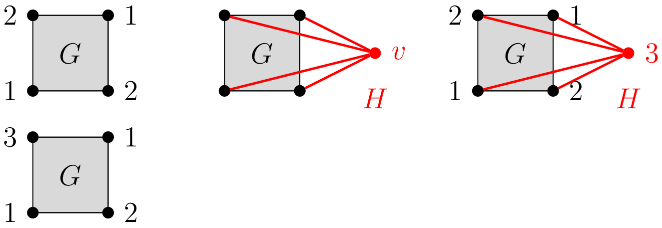

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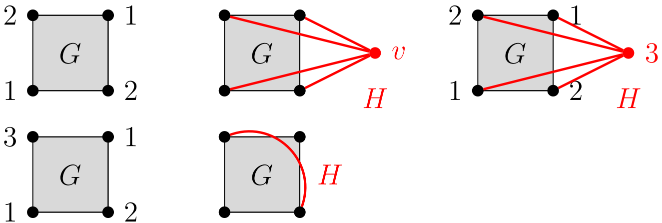

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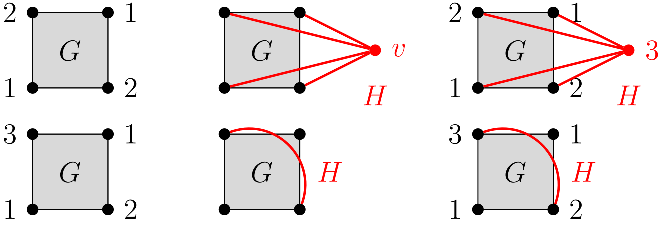

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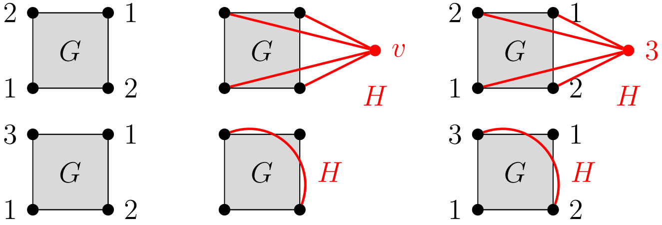

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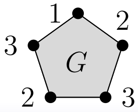

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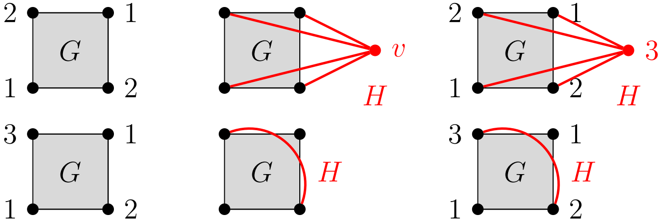

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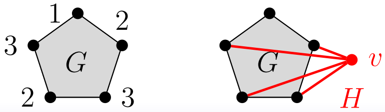

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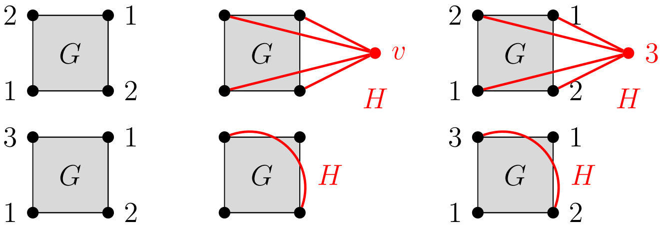

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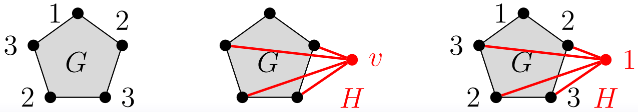
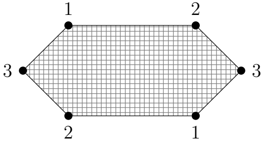


Precolored faces

G is a plane triangle-free graph with a face bounded by a cycle C

- if $|C| \leq 5$, any precoloring of C extends
- if $|C| = 6$, any precoloring of C extends unless G contains



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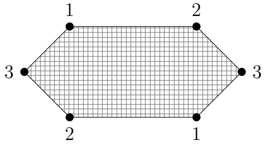

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- $|C| = 7$ Aksenov, Borodin, Glebov '04
- $|C| = 8$ Dvořák, L. '15
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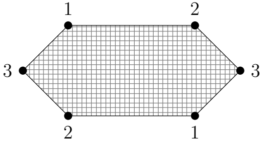

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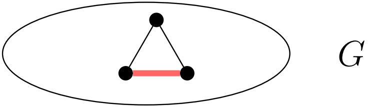
Girth 5 completely solved by Dvořák, Kawarabayashi '11
Girth 4 by Dvořák and Pekárek

Some triangles?

Theorem (Grötzsch '59)

Every planar triangle-free graph is 3-colorable.

We already showed one triangle!



Removing one edge of triangle results in triangle-free G.

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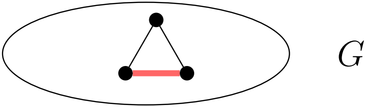

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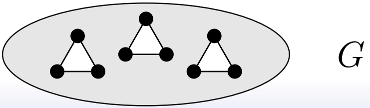
Theorem (Grünbaum '63; Aksenov '74; Borodin '97)

Let G be a planar graph containing at most three triangles. Then G is 3-colorable.



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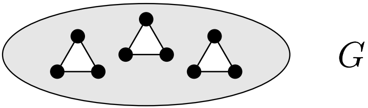
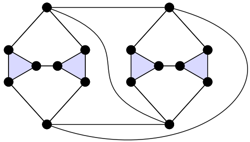


We can simplify the proof.

Question: What about four triangles?
Call 4-critical plane graph with four triangles a $4,4$-graph.

Planar graphs with four triangles?

Havel '69 found

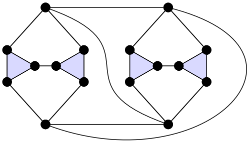


Problem (Sachs '72)

Let G be a $4, 4$-graph. Can the triangles be partitioned into two pairs so that in each pair the distance between the triangles is less than two?

Planar graphs with four triangles?

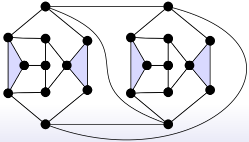
Havel '69 found



Problem (Sachs '72)

Let G be a $4, 4$-graph. Can the triangles be partitioned into two pairs so that in each pair the distance between the triangles is less than two?

No! Aksenov and Mel'nikov '78,'80, Two infinite series of $4, 4$-graphs.

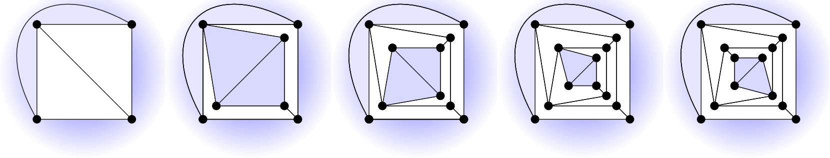


Problem (Erdős '90)

Describe $4,4$*-graphs.*

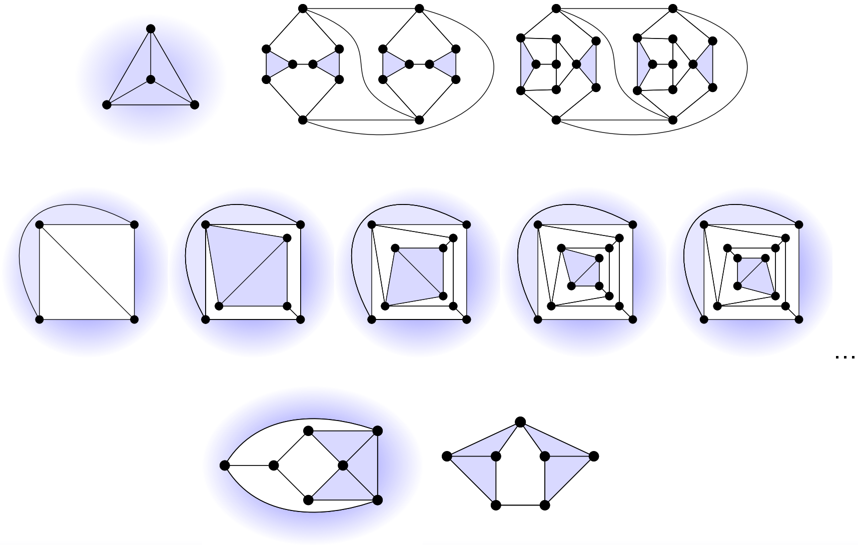
Borodin '97 - at least 15 infinite families of $4,4$-graphs (all with 4-faces)

Thomas and Walls '04 - Infinite family $\mathcal{TW}$ of $4,4$-graphs with no 4-faces.



...

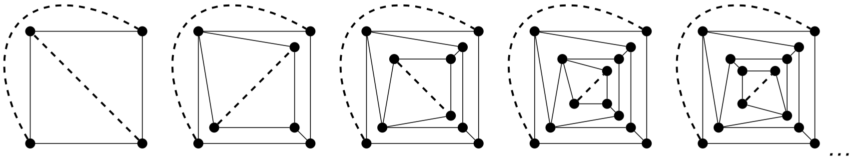
Some known 4, 4-critical planar graphs



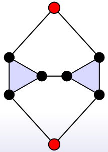
No 4-faces

Theorem (Borodin, Dvořák, Kostochka, L., Yancey '14)

All plane $4,4$-graphs with no 4-faces can be obtained from the Thomas-Walls sequence



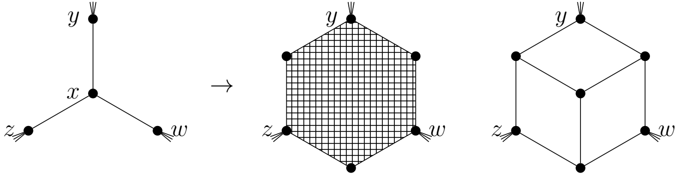
by replacing dashed edges by edges or the gadget (Havel's quaziedge)



With 4-faces

Theorem (Borodin, Dvořák, Kostochka, L., Yancey '14)
All plane $4,4$-graphs with no 4-faces are precisely graphs in $\mathcal{C}$.

Theorem (Borodin, Dvořák, Kostochka, L., Yancey '14)
Every $4,4$-graph can be obtained from $G \in \mathcal{C}$ by expanding some vertices of degree 3.



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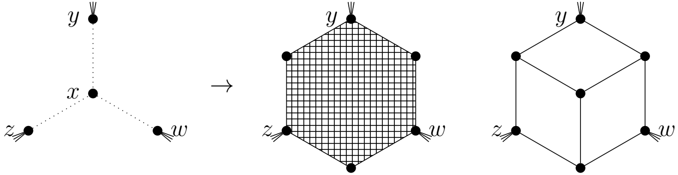

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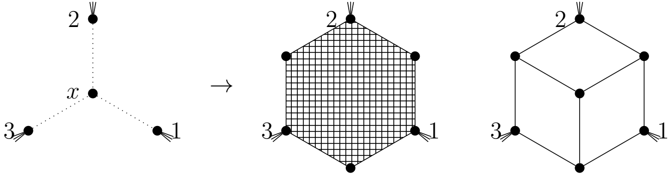

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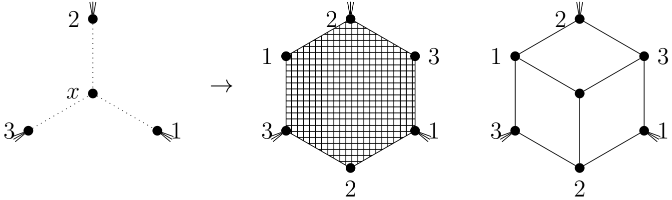
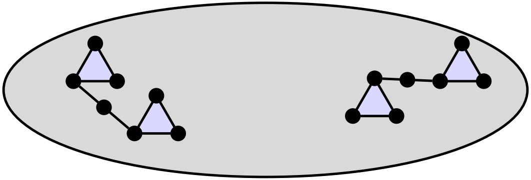


Problem (Sachs '72)

Let G be a $4,4$-graph. Can the triangles be partitioned into two pairs so that in each pair the distance between the triangles is less than two?

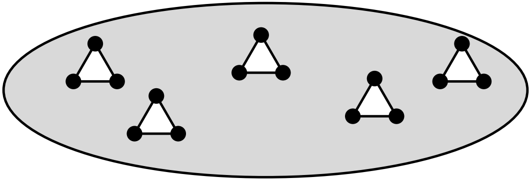
Theorem (Borodin, Dvořák, Kostochka, L., Yancey '14)

Triangles in a $4,4$-graph can be partitioned into two pairs so that in each pair the distance between the triangles is at most two.



Conjecture (Havel '70)

There exists $d > 0$ so that if G is planar with mutual distance of triangles $\geq d$, then G is 3-colorable.



Proved by Dvořák, Kráľ, Thomas.

Theorem (Dvořák, Kráľ, Thomas)

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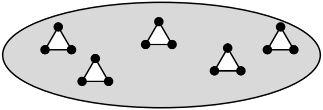

Theorem (Dvořák, Král', Thomas)

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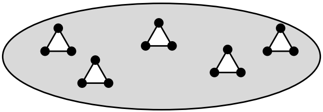
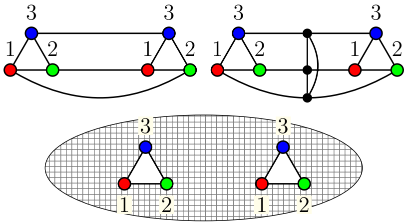


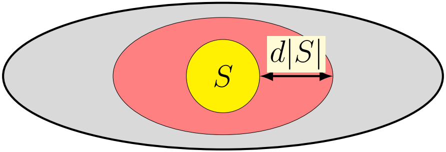
But triangles cannot be precolored.



Winding number in quadrangulation.

THEOREM (DVOŘÁK, KRÁĽ, THOMAS)

There exists $d > 0$ so that if G is planar of girth 5, $S \subset V(G)$ and a precoloring of S extends to all vertices at distance $\leq d|S|$ from S, then it extends to a 3-coloring of G.



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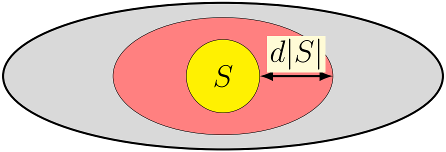
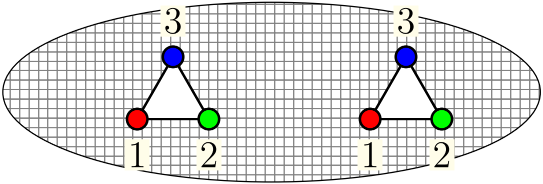
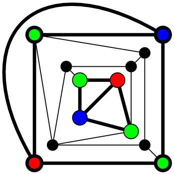


Girth 5 needed:

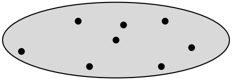




Postle: distance $d = 100$ is sufficient.

Conjecture (Dvořák, Král', Thomas)

There exists $d > 0$ so that if G is planar of girth 4, $S \subset V(G)$ and $\forall u, v \in S\ dist(u, v) \geq d$, then any precoloring of S extends to a 3-coloring of G.



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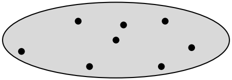

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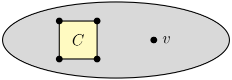

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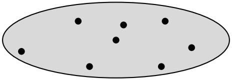

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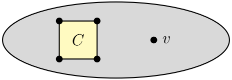

Theorem (Dvořák, Kráľ, Thomas)

Second conjecture implies the first one.

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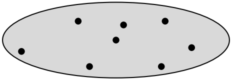

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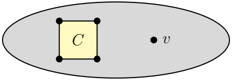

Theorem (Dvořák, Kráľ, Thomas)

Second conjecture implies the first one.

- We prove the second conjecture

Theorem (Dvořák, L.)

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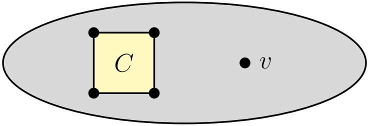

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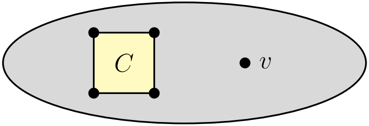
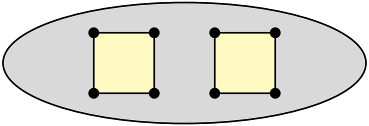


Characterize when a precoloring of two 4-cycles extend.

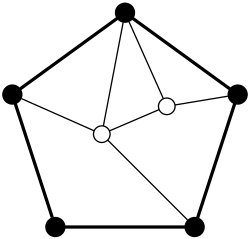


Definition - S-critical graph

For a graph $G = (V, E)$, $S \subset G$

G is an *S-critical graph* for 3-coloring if for every H such that $S \subset H \subset G$ there exists a 3-coloring φ of S where

- φ extends to a 3-coloring of H
- φ does not extend to a 3-coloring of G.



Note that $\emptyset$-critical graph is 4-critical.

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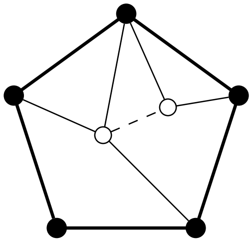

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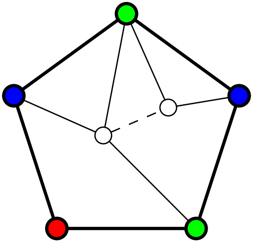

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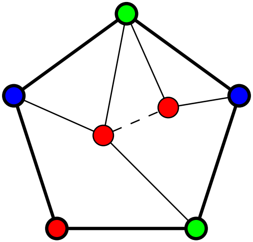
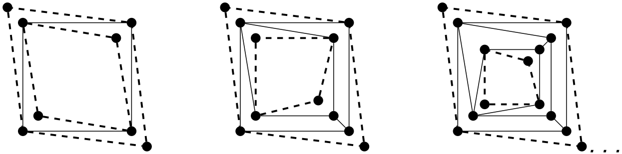


Note that $\emptyset$-critical graph is 4-critical.

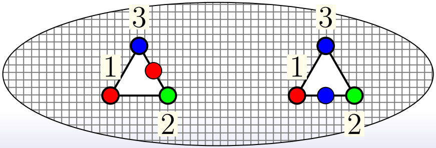
THEOREM (DVOŘÁK, L.)

There exists $d > 0$ so that if G is a plane graph with two 4-faces C_1 and C_2 in distance $\geq d$ and all triangles in G are disjoint, non-contractible, and G is (C_1, C_2)-critical, then

- *G is obtained from framed patched Thomas-Walls, or*

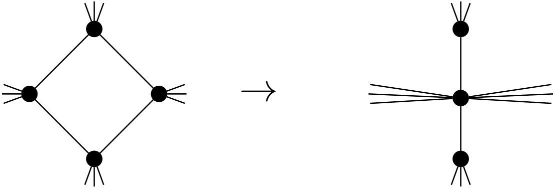


- *G is a near $3, 3$-quadrangulation.*

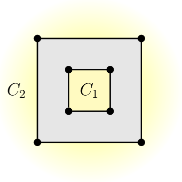


Proof sketch

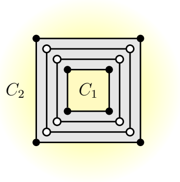
Main tool - collapse 4-faces



Few separating ≤ 4-cycles



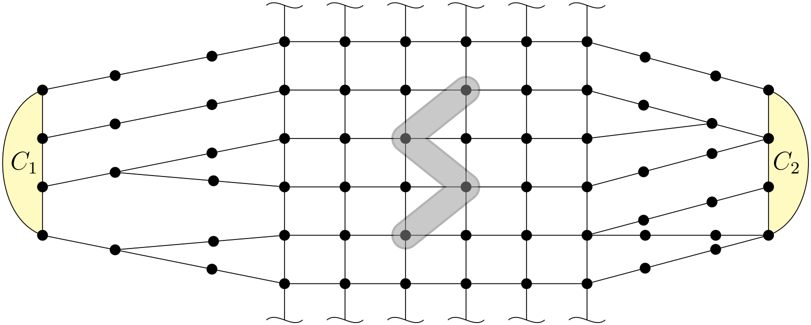
Many separating ≤ 4-cycles



Proof sketch - creating separating ≤ 4-cycles

Lemma

If C_1 and C_2 have distance $\geq d$, then it is possible to create a separating 4-cycle and decrease d by one.



Proof sketch - creating separating ≤ 4-cycles

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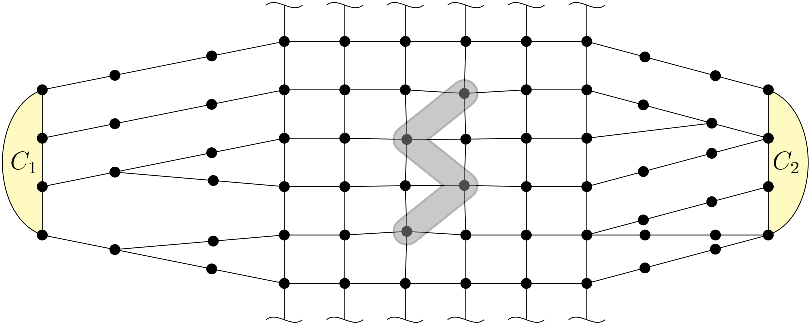

LEMMA

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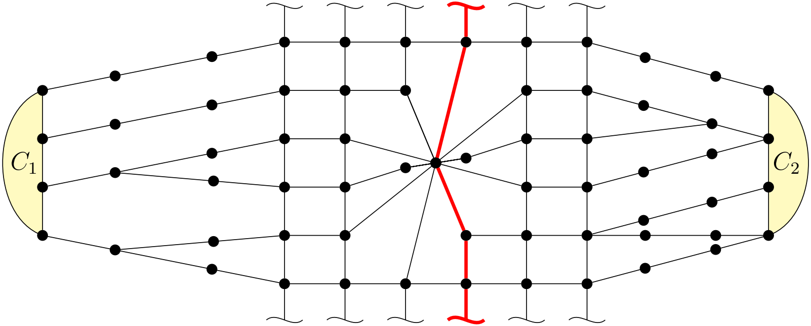
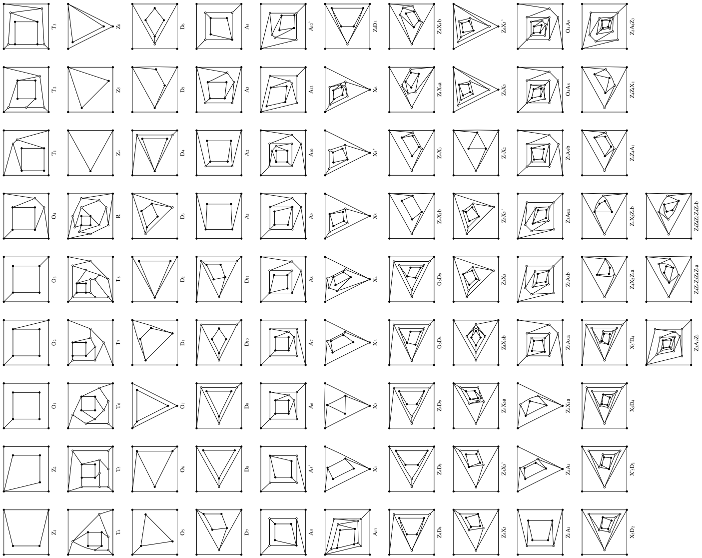


Proof sketch - many separating ≤ 4-cycles

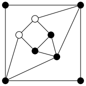
- collapse 4-faces without destroying separating ≤ 4-cycles

- describe *basic* graphs between separating ≤ 4-cycles
  - with 4-faces (21 graphs)
  - without 4-faces (94 graphs)
- gluing of at least ≥ 1056 (or ≥ 40 with computer) basic graphs
  - extends any precoloring of outer cycles, or
  - is Thomas-Walls, or
  - is an almost $3, 3$-quadrangulation.

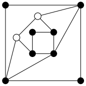
Basic graphs without 4-faces



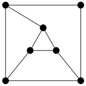
Basic graphs with 4-faces



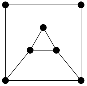
EV_1



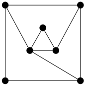
EV_2



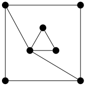
T_1



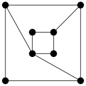
T'_1



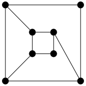
T_2



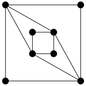
T'_2



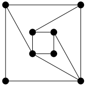
S_1



S_2



Q_1



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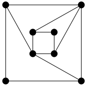

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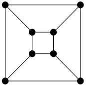
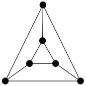
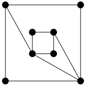


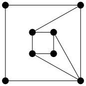
Q_4



Q_5



X_1



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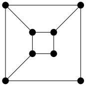
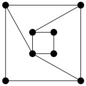


X_3



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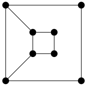

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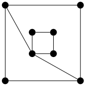
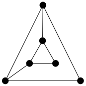
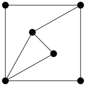


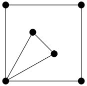
X_6



X_7



J_1

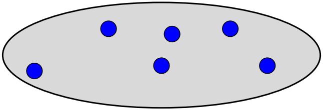


I_4

More consequences

Corollary (Dvořák, L.)

If G is an n-vertex triangle-free planar graph with maximum degree Δ, then G has at least $\left(3^{1/\Delta^D}\right)^n$ distinct colorings, where D is constant.



Pick n/Δ^D vertices S of G in mutual distance $\geq D$.
All $3^{|S|}$ precolorings of S extend to different 3-colorings of G.

More consequences

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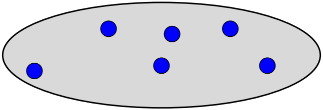
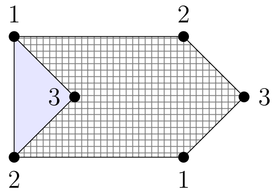


Pick n/Δ^D vertices S of G in mutual distance $\geq D$.
All $3^{|S|}$ precolorings of S extend to different 3-colorings of G.

More consequences

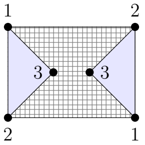
Let G be a plane graph with a face C and t triangles.

- If $|C| = 4$ and $t \leq 1$, then any precoloring of C extends.
- If $|C| = 5$ and $t \leq 1$, then the only C-critical graph is:

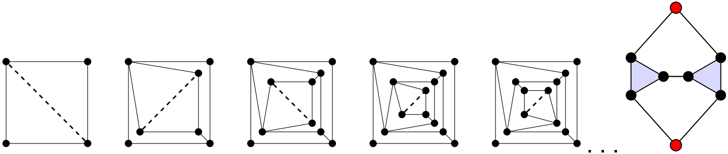


Theorem (Dvořák, L.)

Let G be a plane graph with a 4-face C and 2 triangles. If G is C-critical, then G is



or a framed patched Thomas-Walls, where the dashed edge is a normal edge or Havel's quaziedge.



(C is the outer face, vertices of degree 2 have different colors)

Thank you for your attention!