Decomposing Random d-regular Graphs Into Stars

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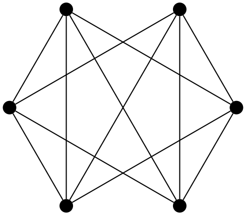
EXtremal Combinatorics at ILLinois 3
Aug 10, 2016

Theorem (Delcourt and Postle 2015+)

If $3|n$ *then a random 4-regular graph on n vertices has a claw decomposition asymptotically almost surely (a.a.s.).*



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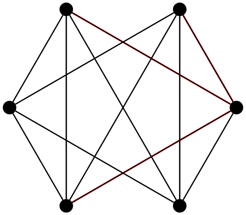

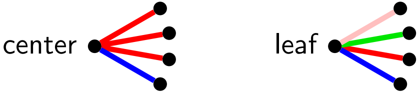

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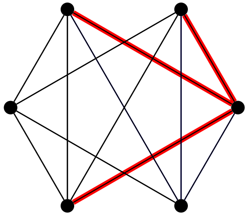

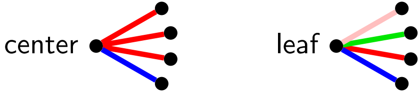

Question (Barát and Thomassen)

Does every 4-edge-connected 4-regular graph have a claw decomposition?

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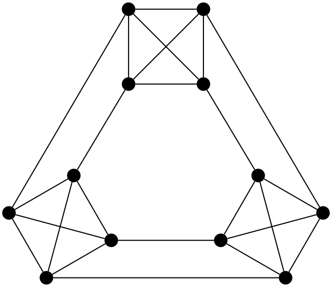
No!



8 center vertices, 4 leaf vertices $\rightarrow$ adjacent leaves

Barát and Thomassen's Conjecture

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If G is a planar 4-edge-connected, 4-regular graph such that $3|e(G)$*, then G has a claw decomposition.*

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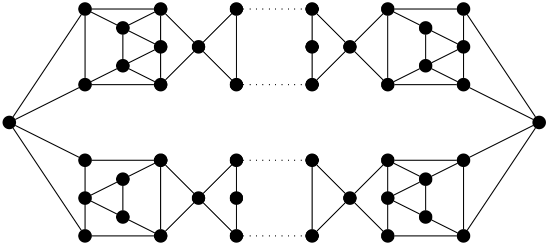
Counterexample Lai 2007



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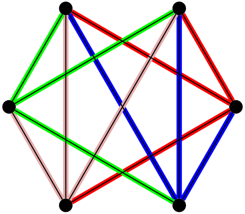

Theorem (Bollobás 1981, Wormald 1981)

Random d-regular graph is d-edge-connected a.a.s..

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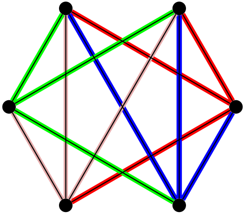

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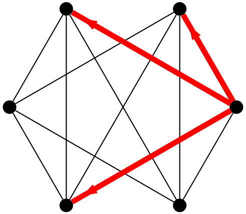

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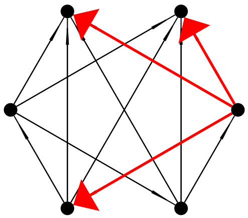

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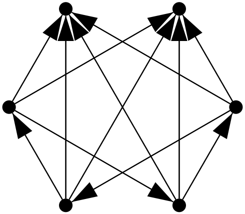

Theorem (Delcourt and Postle 2015+; proved)

If $3|n$ then a random 4-regular graph on n vertices has an orientation in which every outdegree is either 3 or 0 a.a.s..

Conjecture (Tutte 1966; equivalent form)

Every 4-edge connected 5-regular graph has an edge orientation in which every outdegree is either 4 or 1.

Theorem (Pralat and Wormald 2015+)

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Conjecture (Jaeger 1988; equivalent form)

Every $4p$-edge connected $(4p+1)$-regular graph has an edge orientation in which every outdegree is either $3p+1$ or p.

Theorem (Alon and Pralat 2011)

For $p > p_0$, a random $(4p+1)$-regular graph has an orientation in which every outdegree is either $3p+1$ or p.

Theorem (Delcourt and Postle 2015+)

If $3|n$ then a random 4-regular graph on n vertices has a claw decomposition (orientation with outdegrees 3 or 0) a.a.s..

Question: What about d-regular graphs and S_k (star with k leaves)?

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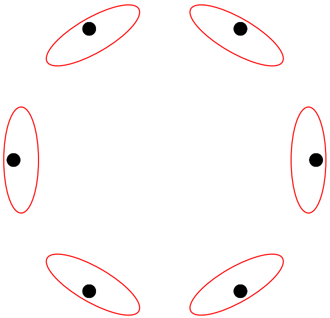
Try: random 4-regular gives orientation with outdegrees 3 or 0....

Random d-regular Graphs and Pairings $\mathcal{P}_{n,d}$



Pairing Model (Bollobás 1980)

1. *Begin with n vertices.*

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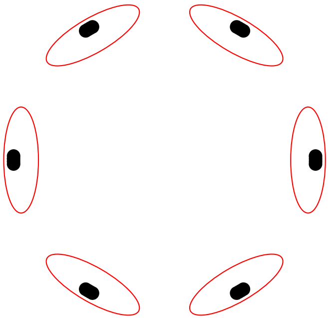

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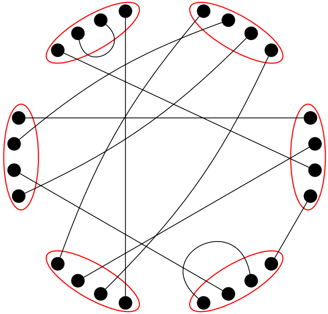

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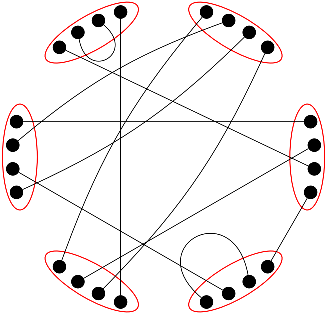

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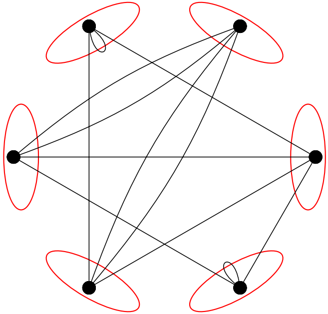

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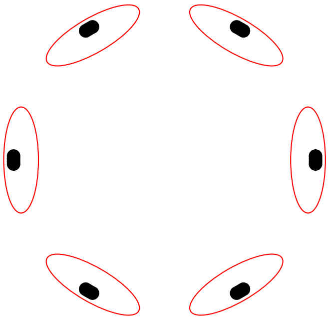

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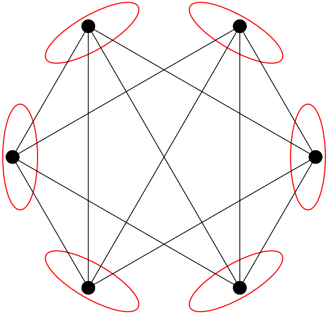

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$P(\text{simple}) \sim \exp\left(\frac{1-d^2}{4}\right)$ for d fixed and $n \to \infty$

Event true a.a.s in $\mathcal{P}_{n,d}$, $\Rightarrow$ true a.a.s. in random d-regular graphs.

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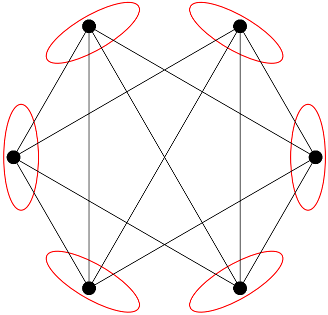

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Janson: Event true a.a.s. in $\mathcal{P}_{n,d}$, $\Rightarrow$ true a.a.s. in random d-regular multigraphs.

Really happening:

THEOREM (DELCOURT AND POSTLE 2015+)

If $3|n$ then a random multigraph from pairing in $\mathcal{P}_{n,4}$ has an orientation with outdegrees 3 or 0 a.a.s..

$Y = Y(n) := \#\ \{3, 0\}$-orientations of a random element of $\mathcal{P}_{n,4}$.
First try: compute $\mathbb{E}[Y]$, $\mathbb{E}[Y^2]$, show

$$\frac{\mathbb{E}[Y^2]}{\mathbb{E}[Y]^2} \to 0.$$

Luke and Michelle

Counting Orientations Using Signatures

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When counting $\mathbb{E}[Y]$:

- for every pairing count # of orientations
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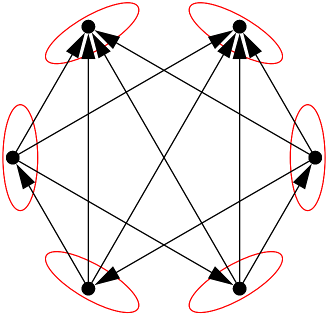

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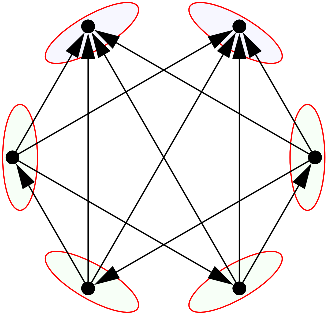

$\binom{n}{2n/3}$ Centers   Leaves (independent set)

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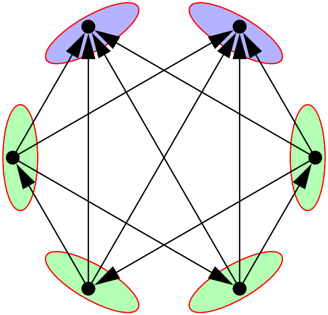

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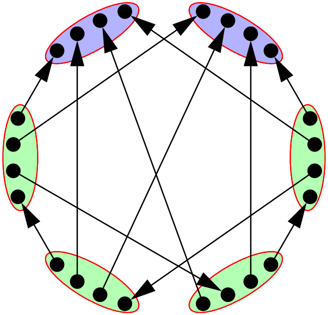

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Special points

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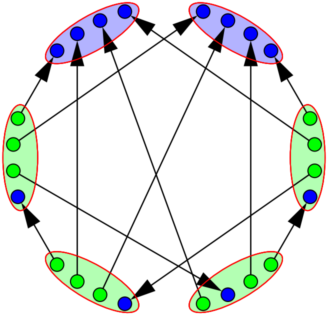

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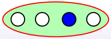
Special points 

Now we have a *signature*.

COMPUTING $\mathbb{E}[Y]$ USING SIGNATURES

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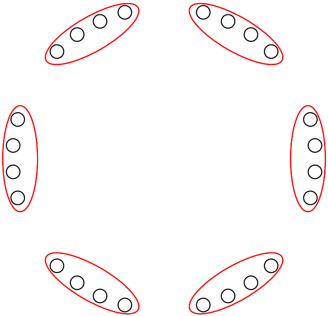
$$\mathbb{E}[Y] = \frac{\binom{n}{2n/3} \cdot 4^{2n/3} \cdot (2n)!}{M(4n)} \sim \frac{3}{\sqrt{2}} \left(\frac{27}{16}\right)^{n/3}.$$



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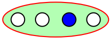
- $4^{2n/3}$ choices of special points for centers. 
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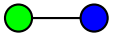

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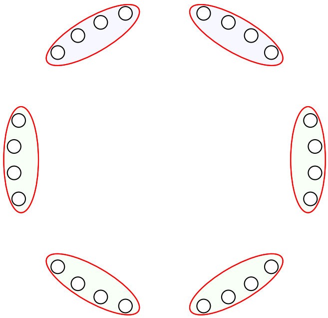

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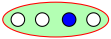
- $4^{2n/3}$ choices of special points for centers. 
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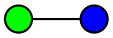
- $\left(\frac{4n}{2}\right)! = (2n)!$ ways to match "out" points to "in" points. 
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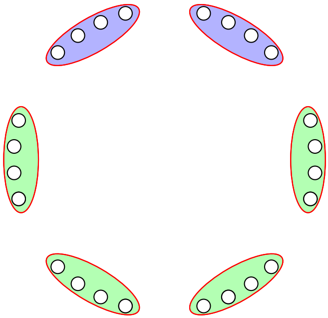

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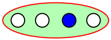

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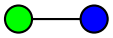

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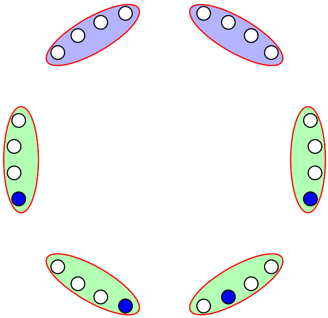

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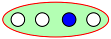
- $4^{2n/3}$ choices of special points for centers. 
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  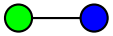

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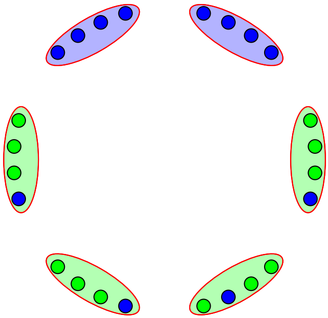

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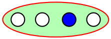
- $4^{2n/3}$ choices of special points for centers. 
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  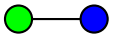
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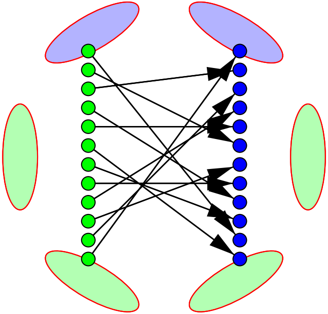
Computing $\mathbb{E}[Y]$ Using Signatures

$Y = Y(n) := \#\ \{3,0\}$-orientations of a random element of $\mathcal{P}_{n,4}$.

$$\mathbb{E}[Y] = \frac{\binom{n}{2n/3} \cdot 4^{2n/3} \cdot (2n)!}{M(4n)} \sim \frac{3}{\sqrt{2}} \left(\frac{27}{16}\right)^{n/3}.$$



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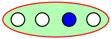
- $4^{2n/3}$ choices of special points for centers. 
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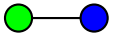
- $\left(\frac{4n}{2}\right)! = (2n)!$ ways to match "out" points to "in" points. 
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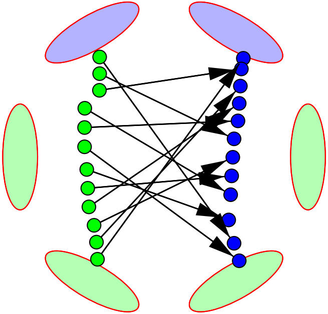

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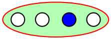
- $4^{2n/3}$ choices of special points for centers. 
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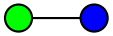
- $\left(\frac{4n}{2}\right)! = (2n)!$ ways to match "out" points to "in" points. 
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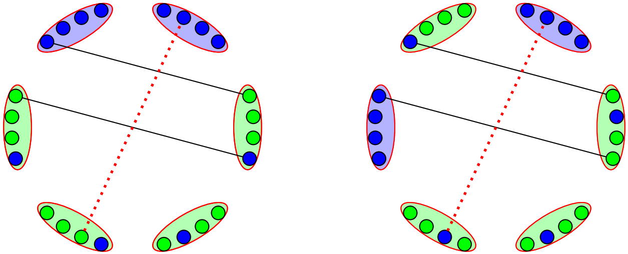
COMPUTING $\mathbb{E}[Y^2]$; $\mathbb{E}[Y] \sim \frac{3}{\sqrt{2}} \left(\frac{27}{16}\right)^{n/3}$

To calculate $\mathbb{E}[Y^2]$, we fix two signatures and see how many configurations they jointly extend to.



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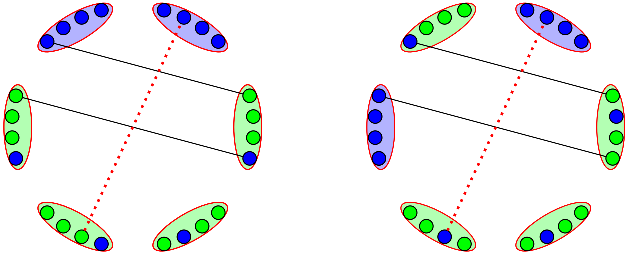

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Hence

$$\frac{\mathbb{E}[Y^2]}{\mathbb{E}[Y]^2} \sim \sqrt{\frac{3}{2}} > 0 \Rightarrow \text{ second moment fails}$$

Main Tool [Robinson and Wormald 1992]

Let $\lambda_j > 0$ and $\delta_j > -1$ be real, $j \geq 1$. Suppose for each n there are non-negative random variables $X_j = X_j(n)$, $j \geq 1$, and $Y = Y(n)$ defined on the same probability space such that X_j is integer valued and $\mathbb{E}[Y] > 0$ (for n sufficiently large). Furthermore, suppose

1. For each $j \geq 1$, $X_1, X_2, \ldots, X_j$ are asymptotically independent Poisson random variables with
$$\mathbb{E}[X_i] \to \lambda_i, \text{ for all } i \in [j];$$
2. $$\frac{\mathbb{E}\left[Y[X_1]_{\ell_1} \ldots [X_j]_{\ell_j}\right]}{\mathbb{E}[Y]} \to \prod_{i=1}^{j} \left(\lambda_i \left(1 + \delta_i\right)\right)^{\ell_i}$$
for any fixed $\ell_1, \ldots, \ell_j$ where $[X]_\ell$ is the falling factorial;
3. $\sum_i \lambda_i \delta_i^2 < \infty$;
4. $$\frac{\mathbb{E}[Y^2]}{\mathbb{E}[Y]^2} \leq \exp\left(\sum_i \lambda_i \delta_i^2\right) + o(1) \text{ as } n \to \infty.$$

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This works for 4-regular and orientations with outdegree $\in \{0, 3\}$.
Now we trying $\{0, k\}$ in d-regular.

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Let X_i denote the number of cycles of length i in the random multigraph resulting from a pairing in $\mathcal{P}_{n,d}$.

Theorem (Bollobás 1980)

For $j \geq 1$, $X_1, \ldots, X_j$ are asymptotically independent Poisson random variables with

$$\mathbb{E}[X_i] \to \lambda_i := \frac{(d-1)^i}{2 \cdot i}$$

for all $i \in [j]$.

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$$\frac{\mathbb{E}\left[Y[X_1]_{\ell_1}\dots[X_j]_{\ell_j}\right]}{\mathbb{E}[Y]} \to \prod_{i=1}^{j}\left(\lambda_i\left(1+\delta_i\right)\right)^{\ell_i}$$

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We show that for each $j \geq 1$,

$$\frac{\mathbb{E}\left[YX_j\right]}{\mathbb{E}[Y]} \to \lambda_j\left(1+\delta_j\right)$$

$$\mathbb{E}[YX_j] = \mathbb{E}[Y]\cdot\lambda_j\left(1+\left(\frac{d-2k+1}{d-1}\right)^j\right);\ \delta_j = \left(\frac{d-2k+1}{d-1}\right)^j$$

3. $\sum_i \lambda_i \delta_i^2 < \infty$;

$$\lambda_i = \frac{(d-1)^i}{2i} \qquad\qquad \delta_i = \left(\frac{d-2k+1}{d-1}\right)^i$$

$$\sum_i \lambda_i \delta_i^2 = -\frac{1}{2}\ln\left(\frac{4k-2-d-(2k-d)^2}{d-1}\right) < \infty$$

Hence $k < \frac{d}{2} + \frac{0.5+\sqrt{d+1}}{2}$.

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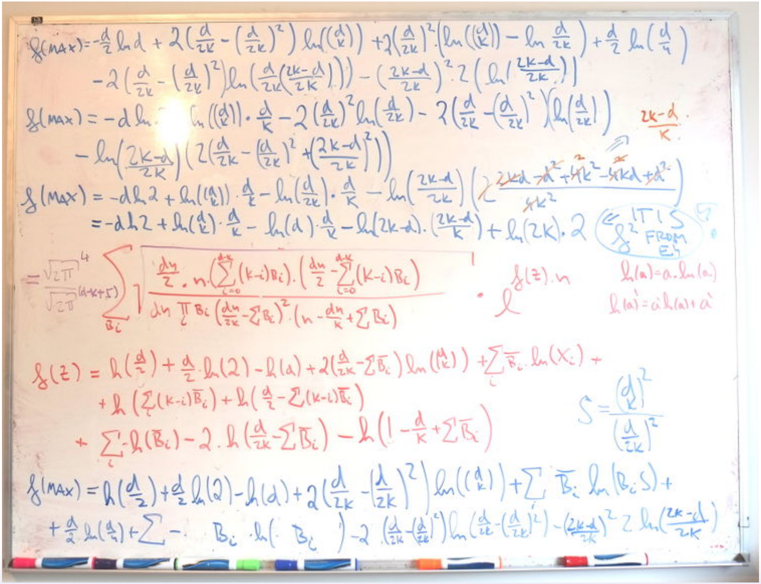
Guess cannot be true!

It MUST hold $k \leq \frac{d}{2} + 2 \log d$.

Dealing with $\mathbb{E}[Y^2]$



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THEOREM (DELCOURT, L., POSTLE)

A random d-regular graph on n vertices has orientation with outdegrees in $\{0, k\}$ a.a.s. (if $k|dn/2$, $k > d/2$) for

- *small d and $k < \frac{d}{2} + 2\log d - c$ (with machine d up to 50)*
- *larger d and k up to $\frac{d}{2} + \Omega(\log d)$?*

Theorem (Delcourt, L., Postle)

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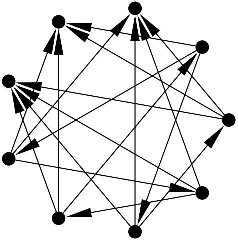

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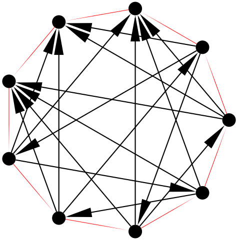

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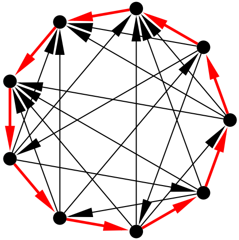

$\mathcal{G}_{n,d} \oplus \mathcal{H}_n \approx \mathcal{G}_{n,d+2}$ if $d \geq 1$ (a.a.s. holds in both)

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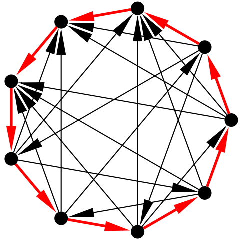

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$\{0, k\}$ in $\mathcal{G}_{n,d}$ gives $\{j, k+j\}$ in $\mathcal{G}_{n,d+2j}$

 Happy Birthday! 

Conjecture (Jaeger 1988; equivalent form)

Every $4p$-edge connected $(4p+1)$-regular graph has an edge orientation in which every outdegree is either $3p+1$ or p.

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Theorem (Borodin, Kim, Kostochka, West '04)

Every planar graph of girth $\geq \frac{20p-2}{3}$ has a homomorphism to C_{2p+1}.

Happy Birthday!