Independent sets near the lower bound in bounded degree graphs

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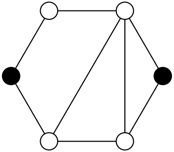
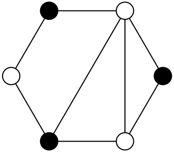
Independent sets

Definition

An *independent set* in a graph G is an induced subgraph with no edges.





$\alpha(G)$ is the size of a maximum independent set in G.
$\omega(G)$ is the size of a maximum clique in G.

Trivial lower bound

$\Delta(G)$ is the maximum degree of G.
n is the number of vertices of G.

If $\Delta(G) \leq \Delta$, then $\alpha(G) \geq \frac{n}{\Delta+1}$.

Trivial lower bound

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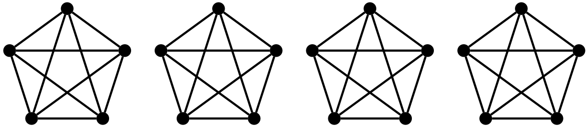
If $\Delta(G) \leq \Delta$, then $\alpha(G) \geq \frac{n}{\Delta+1}$.
(tight)

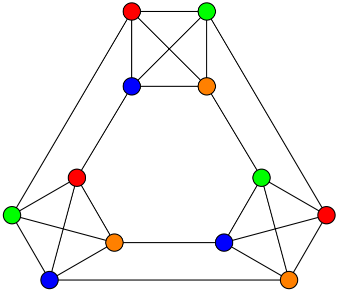


What is $\omega(G) \leq \Delta$?

Forbidding largest cliques

Theorem (Brooks 1941)

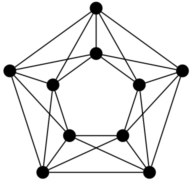
If $\Delta(G) \geq 3$ and $\max(\Delta(G), \omega(G)) \leq \Delta$ then G is Δ-colorable.

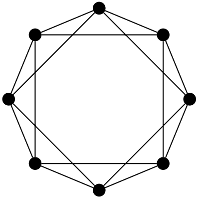


Implies $\alpha(G) \geq \frac{n}{\Delta}$.
Tight.

Theorem (Albertson, Bollobás, Tucker 1976)

If G is connected, $\Delta(G) \le \Delta$ and $\omega(G) \le \Delta - 1$, then $\alpha(G) > \frac{n}{\Delta}$ unless G is one of the following two exceptions:





Related results

Theorem (Albertson, Bollobás, Tucker 1976)

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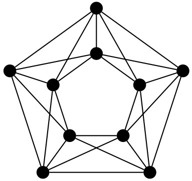
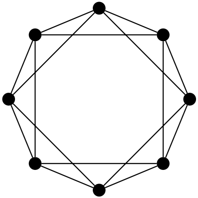


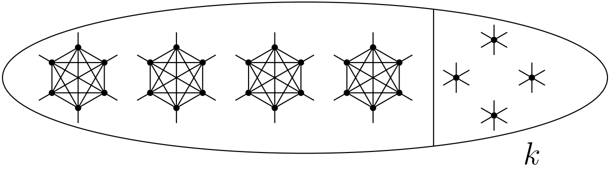


Theorem (King, Lu, Peng 2012)

If G is connected, $\Delta(G) \leq \Delta$ and $\omega(G) \leq \Delta - 1$, then $\alpha(G) > \frac{n}{\Delta - \frac{2}{67}}$ unless G is one of the two exceptions above.

Small surplus k

If G with $\max(\Delta(G), \omega(G)) \leq \Delta$ is



then $\alpha(G) \leq \frac{n-k}{\Delta} + k$.

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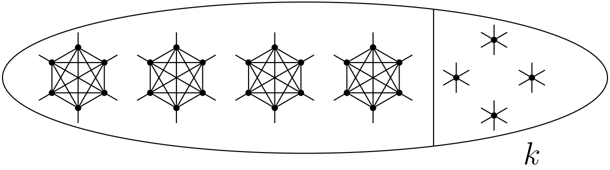
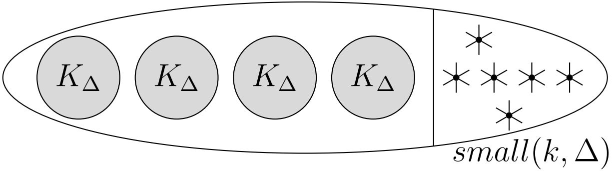


then $\alpha(G) \leq \frac{n-k}{\Delta} + k$.

Problem

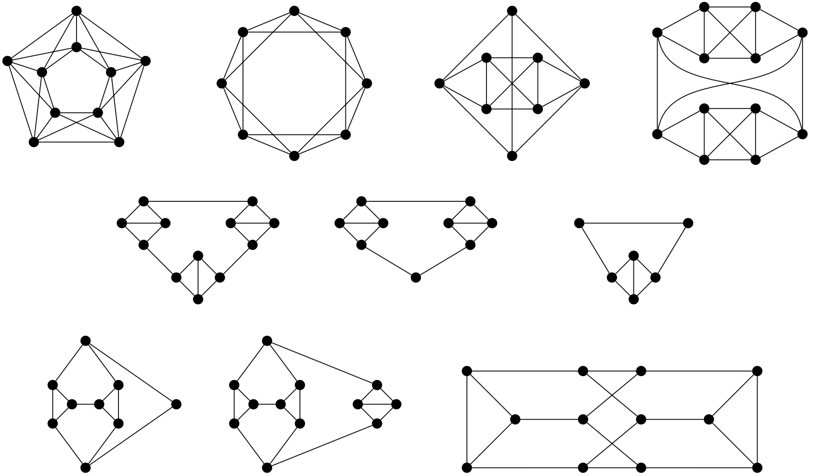
If $\alpha(G) \leq \frac{n}{\Delta} + k$ *and* $\max(\Delta(G), \omega(G)) \leq \Delta$*, does* G *look like*



Are there other candidates for K_Δ?

Δ-TIGHT GRAPHS

A graph is *Δ-tight* if it is K_Δ or one of

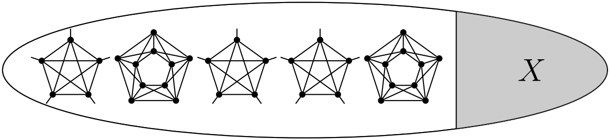


If G is Δ-tight, then $\alpha(G) = \frac{n}{\Delta}$.

Our result

Theorem (Dvořák, L.)

Let $\Delta \geq 3$ and $k \geq 0$.
Let G be an n-vertex graph with $\max(\Delta(G), \omega(G)) \leq \Delta$.
If $\alpha(G) < \frac{n}{\Delta} + k$, then there exists $X \subseteq V(G)$ of size $< 34\Delta^2 k$ such that $G - X$ is Δ-tightly partitioned.



We will try a sketch for $\Delta = 5$.

Δ-free vertices

Definition

A vertex v of G is *Δ-free* if v is not in Δ-tight subgraph.

Many Δ-free vertices $\Rightarrow$ large α

Lemma

If $\max(\Delta(G), \omega(G)) \leq \Delta$ *and* G *has at least* m *vertices that are* Δ*-free then* $\alpha(G) \geq \frac{n}{\Delta} + \frac{1}{34\Delta^2} m$.

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By induction (now only for $\Delta \geq 5$).

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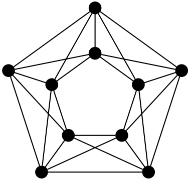
- G contains $C_5 \boxtimes P_2$ (hence $\Delta = 5$)



Let $H = G - C_5 \boxtimes P_2$

$\alpha(G) = \alpha(C_5 \boxtimes P_2) + \alpha(H) = 2 + \frac{n-10}{\Delta} + \frac{1}{34\Delta^2}m = \frac{n}{\Delta} + \frac{1}{34\Delta^2}m$

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- $\omega(G) \leq \Delta - 1$ then $\alpha(G) \geq \frac{n}{\Delta} + \frac{n}{34\Delta^2}$ by King, Lu, Peng.

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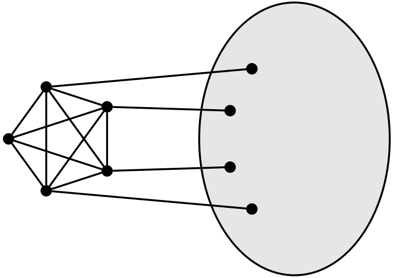

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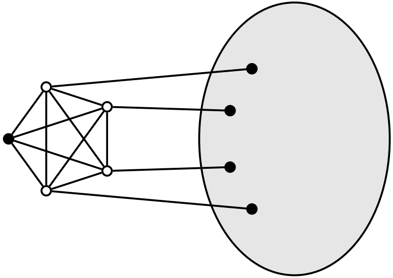

$$\alpha(G) = \alpha(K_\Delta) + \alpha(G - K_\Delta) \geq 1 + \frac{n - \Delta}{\Delta} + \frac{1}{34\Delta^2} m$$
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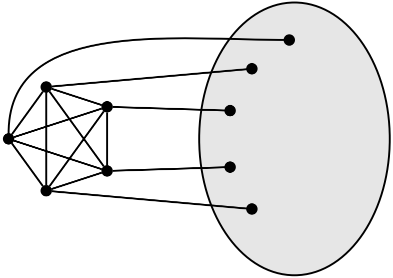

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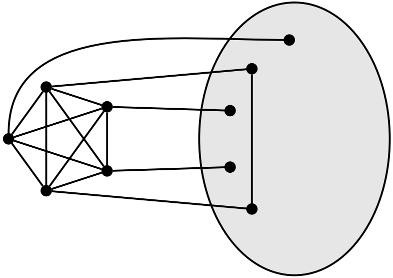

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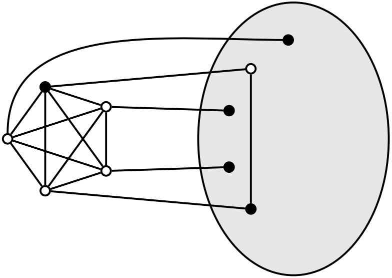

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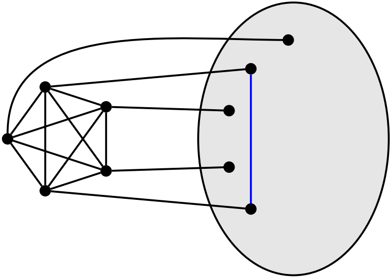

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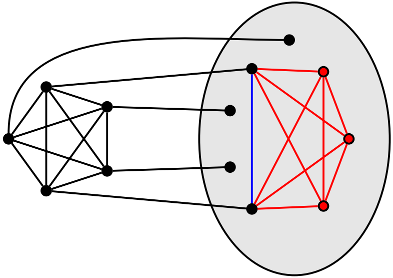

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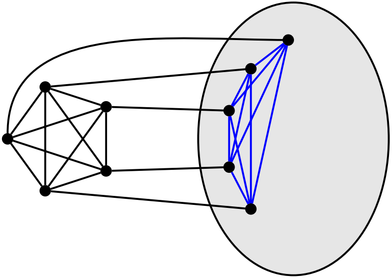

No blue triangles.

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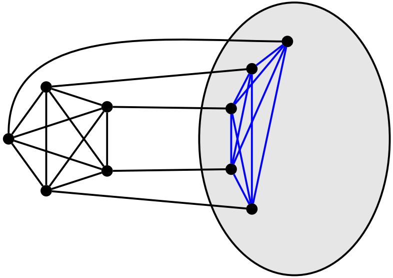

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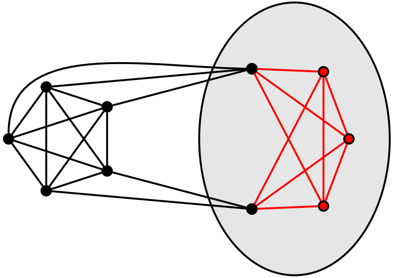


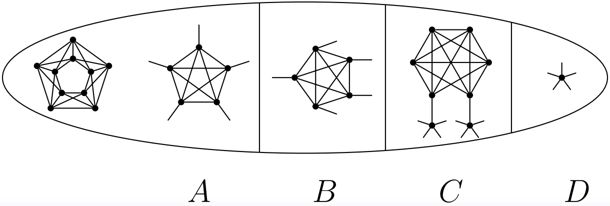
Contradiction with $\Delta(G) \leq \Delta$.

Partition lemma

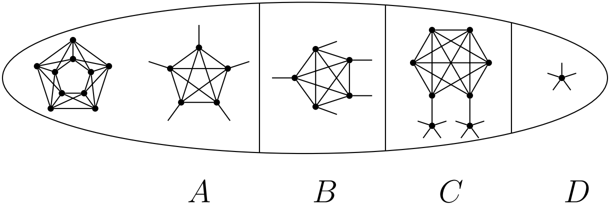
Lemma

An n-vertex graph G with $\max(\Delta(G), \omega(G)) \leq \Delta$ can be partitioned into to sets A, B, C, and D in time $O(\Delta^2 n)$, such that

- $G[A]$ *is* Δ*-tightly partitioned,*
- $G[B]$ *is* K_Δ*-partitioned and* $|B| \leq 3\Delta(|C| + |D|)$,
- C *is* Δ*-profitably nibbled,*
- D *is* Δ*-free in* $G - C$*, and*
- $\alpha(G) = \alpha(G[B \cup C \cup D]) + |A|/\Delta$.



Counting lemma

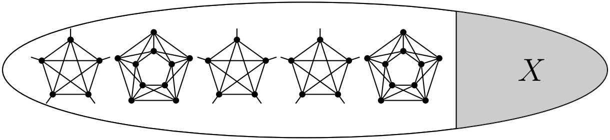


Lemma

If $\alpha(G) < \Delta/n + k$, *then* $|C| + |D| < 34\Delta^2 k$.

Theorem (Dvořák, L.)

If $\alpha(G) < n/\Delta + k$, *then exsits* $X \subseteq V(G)$ *of size* $< 34\Delta^2 k$ *such that* $G - X$ *is* Δ*-tightly partitioned.*



Previous Lemma:
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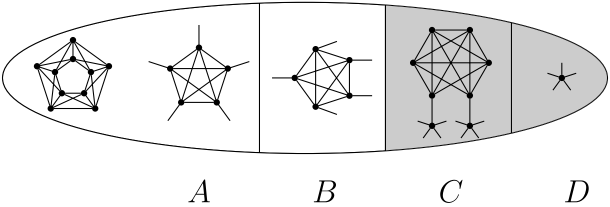

Put $X = C \cup D$.

Computational Consequences

Computing $\alpha(G)$ is NP-complete.

If $\alpha(G) < n/\Delta + k$, can you compute $\alpha(G)$ efficiently?

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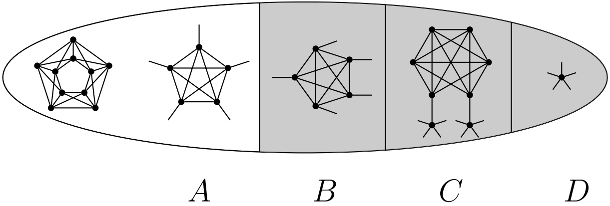
YES!

- Find partition with $|B \cup C \cup D| < 114\Delta^3 k$ in time $O(\Delta^2 n)$.



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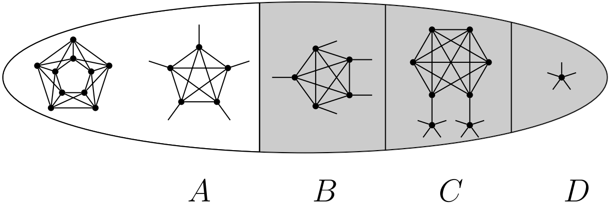

- Compute $\alpha(B \cup C \cup D)$ in time $2^{O(\Delta^3 k)}$.

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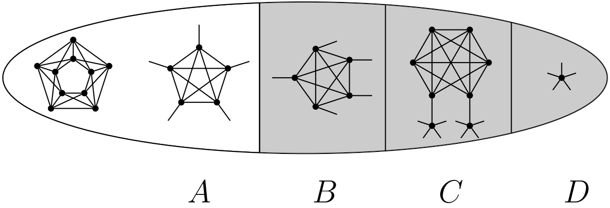

- Compute $\alpha(B \cup C \cup D)$ in time $2^{O(\Delta^3 k)}$.
- Result is $\alpha(B \cup C \cup D) + \frac{|A|}{\Delta}$.

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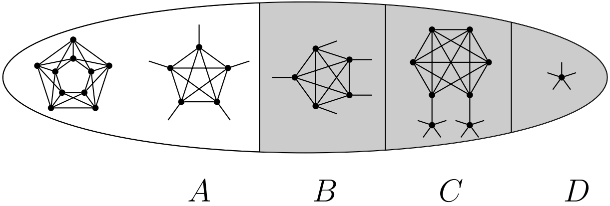

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- Result is $\alpha(B \cup C \cup D) + \frac{|A|}{\Delta}$.
- Total time is $2^{O(\Delta^3 k)} + O(\Delta^2 n)$. Efficient if Δ and k fixed.

Thank you for your attention!