

SMALL RAMSEY NUMBERS

Bernard Lidický Florian Pfender

Iowa State University

University of Colorado Denver



**2017 MMMM Combinatorics
Graduate Students Workshop**

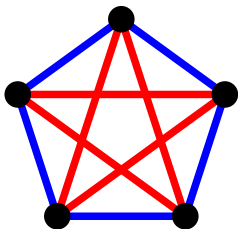
**April 28–30, 2017
University of Minnesota Duluth**

http://www.d.umn.edu/~dfroncek/MMMM_2017/MMMM_2017.pdf

DEFINITION

$R(G_1, G_2, \dots, G_k)$ is the smallest integer n such that any k -edge coloring of K_n contains a copy of G_i in color i for some $1 \leq i \leq k$.

$$R(K_3, K_3) > 5$$

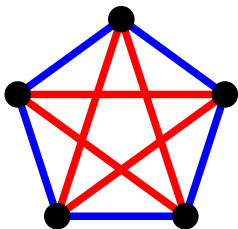


$$R(K_3, K_3) \leq 6$$

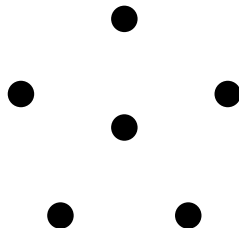
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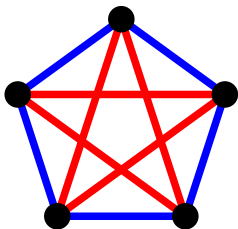
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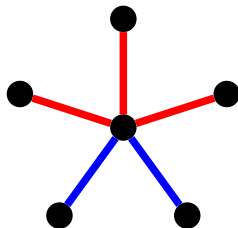
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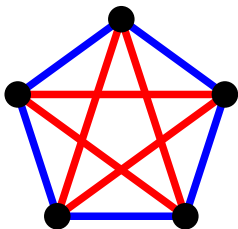
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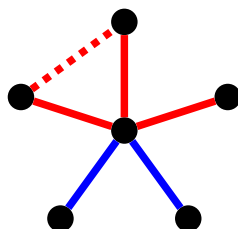
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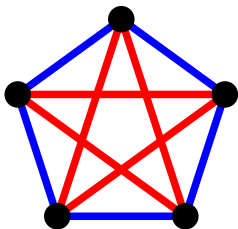
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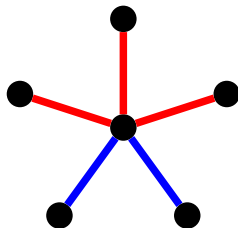
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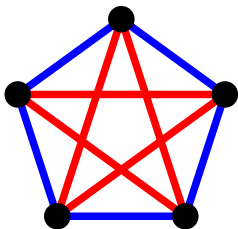
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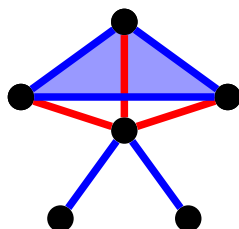
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THEOREM (RAMSEY 1930)

$R(K_m, K_n)$ is finite.



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Questions:

- study how $R(G_1, \dots, G_k)$ grows if G_1, \dots, G_k grow (large)
- study $R(G_1, \dots, G_k)$ for fixed G_1, \dots, G_k (small)



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Radziszowski - *Small Ramsey Numbers*

Electronic Journal of Combinatorics - Survey



FLAG ALGEBRAS

Seminal paper:

Razborov, Flag Algebras, *Journal of Symbolic Logic* **72** (2007), 1239–1282.

David P. Robbins Prize by AMS for Razborov in 2013



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EXAMPLE (GOODMAN, RAZBOROV)

If the density of edges is at least $\rho > 0$, what is the minimum density of triangles?

- designed to attack extremal problems.
- works well if constraints as well as desired value can be computed by checking small subgraphs (or average over small subgraphs)
- the results are in limit (very large graphs)

APPLICATIONS (INCOMPLETE LIST)

| Authors | Year | Application/Result |
|---|------|--|
| Razborov | 2008 | edge density vs. triangle density |
| Hladký, Král', Norin | 2009 | Bounds for the Caccetta-Haggvist conjecture |
| Razborov | 2010 | 3-hypergraphs with forbidden 4-vertex configurations |
| Hatami, Hladký, Král', Norine, Razborov / Grzesik | 2011 | Erdős Pentagon problem |
| Hatami, Hladký, Král', Norin, Razborov | 2012 | Non-Three-Colourable Common Graphs Exist |
| Balogh, Hu, Lidický, Liu / Baber | 2012 | 4-cycles in hypercubes |
| Reiher | 2012 | edge density vs. clique density |
| Das, Huang, Ma, Naves, Sudakov | 2013 | minimum number of k -cliques |
| Baber, Talbot | 2013 | A Solution to the $2/3$ Conjecture |
| Falgas-Ravry, Vaughan | 2013 | Turán density of many 3-graphs |
| Cummings, Král', Pfender, Sperfeld, Treglown, Young | 2013 | Monochromatic triangles in 3-edge colored graphs |
| Kramer, Martin, Young | 2013 | Boolean lattice |
| Balogh, Hu, Lidický, Pikhurko, Udvari, Volec | 2013 | Monotone permutations |
| Norin, Zwols | 2013 | New bound on Zarankiewicz's conjecture |
| Huang, Linial, Naves, Peled, Sudakov | 2014 | 3-local profiles of graphs |
| Balogh, Hu, Lidický, Pfender, Volec, Young | 2014 | Rainbow triangles in 3-edge colored graphs |
| Balogh, Hu, Lidický, Pfender | 2014 | Induced density of C_5 |
| Goac, Hubbard, de Verclous, Sérieni, Volec | 2014 | Order type and density of convex subsets |
| Coregliano, Razborov | 2015 | Tournaments |

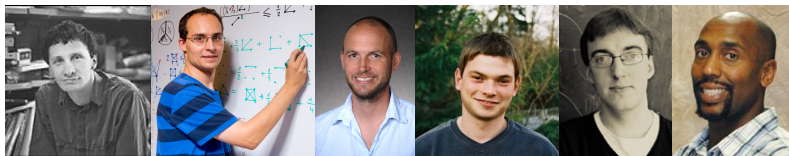
Applications to graphs, oriented graphs, hypergraphs, hypercubes, permutations, crossing number of graphs, order types, geometry, ... Razborov: Flag Algebra: an Interim Report

INSPIRATION

THEOREM (CUMMINGS, KRÁL, PFENDER, SPERFELD, TREGLOWN, YOUNG)

In every 3-edge-colored complete graph on n vertices, there are at least $\frac{1}{25} \binom{n}{3} + o(n^3)$ monochromatic triangles.


$$\text{Green Triangle} + \text{Red Triangle} + \text{Blue Triangle} \geq \frac{1}{25} + o(1)$$



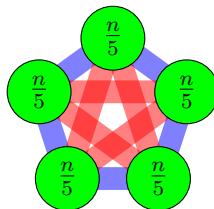
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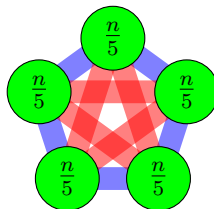
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subject to




$$=$$


$$= 0$$

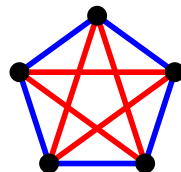
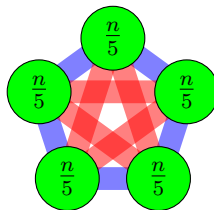
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
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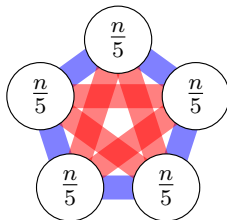
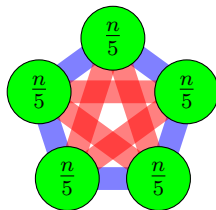
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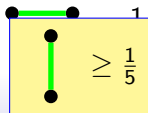
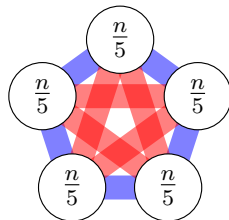
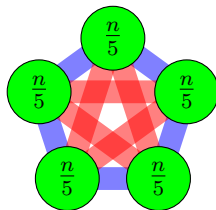
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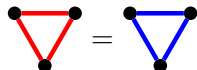
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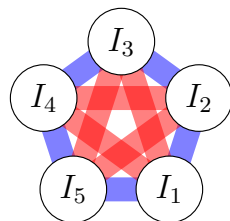
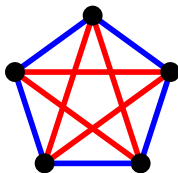


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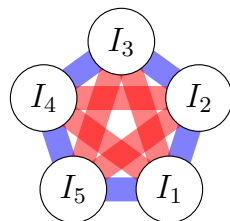
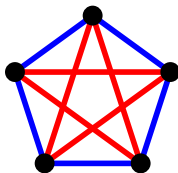
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If a Ramsey graph G has k vertices, then the density of non-edges in any blow-up of G is at least $\frac{1}{k} + o(1)$.

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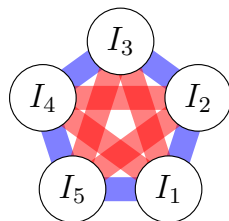
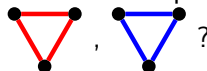
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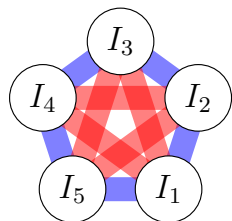
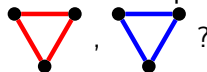
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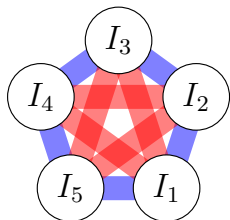
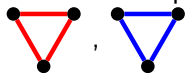


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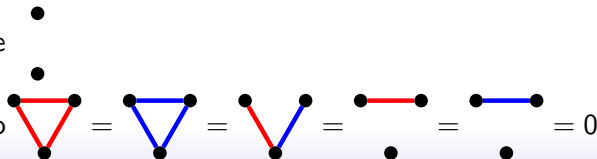


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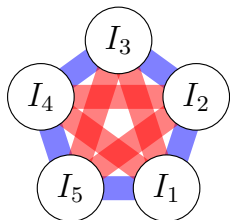
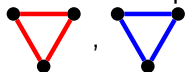
minimize

subject to



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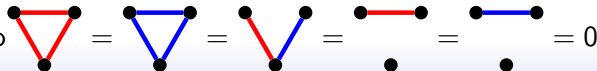
Forbidden subgraphs :



minimize

Flag Algebra question! Easy to modify.

subject to



NEW UPPER BOUNDS (SO FAR)

| Problem | Lower | New upper | Old upper |
|---------------------------|-------|-----------|-----------|
| $R(K_4^-, K_8^-)$ | 29 | 32 | 38 |
| $R(K_4^-, K_9^-)$ | 31 | 46 | 53 |
| $R(K_4, K_7^-)$ | 37 | 49 | 52 |
| $R(K_5^-, K_6^-)$ | 31 | 38 | 39 |
| $R(K_5^-, K_7^-)$ | 40 | 65 | 66 |
| $R(K_5, K_6^-)$ | 43 | 62 | 66 |
| $R(K_5, K_7^-)$ | 58 | 102 | 110 |
| $R(K_6^-, K_7^-)$ | 59 | 124 | 135 |
| $R(K_7, K_4^-)$ | 28 | 29 | 30 |
| $R(K_8, K_4^-)$ | 29 | 39 | 42 |
| $R(K_8, C_5)$ | 29 | 29 | 33 |
| $R(K_9, C_5)$ | 33 | 36 | ?? |
| $R(K_9, C_6)$ | 41 | 41 | ?? |
| $R(K_9, C_7)$ | 49 | 58 | ?? |
| $R(K_{2,2,2}, K_{2,2,2})$ | 30 | 32 | 60? |

| Problem | Lower | New upper | Old upper |
|-----------------------|-------|-----------|-----------|
| $R(K_{3,4}, K_{2,5})$ | | 20 | 21 |
| $R(K_{3,4}, K_{3,3})$ | | 20 | 25 |
| $R(K_{3,4}, K_{3,4})$ | | 25 | 30 |
| $R(K_{3,5}, K_{1,6})$ | 17 | 17 | |
| $R(K_{3,5}, K_{2,4})$ | 16 | 20 | |
| $R(K_{3,5}, K_{2,5})$ | 21 | 23 | |
| $R(K_{3,5}, K_{3,3})$ | | 24 | 28 |
| $R(K_{3,5}, K_{3,4})$ | | 29 | 33 |
| $R(K_{3,5}, K_{3,5})$ | 30 | 33 | 38 |
| $R(K_{4,4}, K_{4,4})$ | 30 | 49 | 62 |
| $R(W_7, W_4)$ | | 21 | |
| $R(W_7, W_5)$ | | 16 | |
| $R(W_7, W_6)$ | | 19 | |
| $R(B_4, B_5)$ | 17 | 19 | 20 |
| $R(B_3, B_6)$ | 17 | 19 | 22 |
| $R(B_5, B_6)$ | 22 | 24 | 26 |



| Problem | Lower | New upper | Old upper |
|-----------------------------|-------|-----------|-----------|
| $R(W_5, K_6)$ | 33 | 36 | |
| $R(W_5, K_7)$ | 43 | 50 | |
| $R(Q_3, Q_3)$ | 13 | 13 | 14 |
| $R(K_3, C_5, C_5)$ | 17 | 17 | 21? |
| $R(K_3, C_4, C_4, C_4)$ | 24 | 29 | ?? |
| $R(K_4, C_4, C_4)$ | 52 | 71 | 72 |
| $R(K_4^-, K_4^-, K_4^-)$ | 28 | 28 | 30 |
| $R(K_3, K_4^-, K_4^-)$ | 21 | 23 | 27 |
| $R(K_4, K_4^-, K_4^-)$ | 33 | 47 | 59 |
| $R(K_4, K_4, K_4^-)$ | 55 | 104 | 113 |
| $R(K_3, K_4, K_4^-)$ | 30 | 40 | 41 |
| $R(K_4^-, K_5^-; 3)$ | 12 | 12 | ?? |
| $R(K_4^-, K_5; 3)$ | 14 | 16 | ?? |
| $R(K_4^-, K_4^-, K_4^-; 3)$ | 13 | 14 | 16 |

$$R(K_4^-, K_5^-; 3) = 12$$

- No lower bound better than 10 was known.

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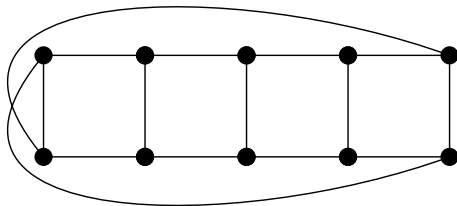
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...and suggest that a Ramsey graph on 11 vertices can only have subgraphs on 8 vertices from a very short list.
- Generating all graphs from this short list is not difficult, and the (unique?) Ramsey graph can be found.



EXAMPLE OF COMPUTATION

LEMMA

$$R(K_3, K_3) \leq 6$$

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Our goal is to show:

$$\begin{array}{c} \bullet \\ \bullet \end{array} > \frac{1}{6} \text{ subject to } \begin{array}{c} \bullet & \bullet \\ & \backslash / \\ & \bullet \end{array} \text{ (red) } = \begin{array}{c} \bullet & \bullet \\ & \backslash / \\ & \bullet \end{array} \text{ (blue) } = \begin{array}{c} \bullet & \bullet \\ & \backslash / \\ & \bullet \end{array} \text{ (split) } = \begin{array}{c} \bullet & \bullet \\ \text{---} & \\ \bullet & \end{array} \text{ (red) } = \begin{array}{c} \bullet & \bullet \\ \text{---} & \\ \bullet & \end{array} \text{ (blue) } = 0$$

EXAMPLE OF COMPUTATION

LEMMA

$$R(K_3, K_3) \leq 6$$

Our goal is to show:

$$\begin{array}{c} \bullet \\ > \frac{1}{6} \text{ subject to } \end{array} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \text{---} \\ \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \text{---} \\ \bullet \end{array} = 0$$

We show perhaps the most complicated proof of the lemma!

Our goal is to show:

$$\begin{array}{c} \bullet \\ \bullet \end{array} > \frac{1}{6} \text{ subject to } \begin{array}{c} \bullet & & \bullet \\ & \diagdown \quad \diagup \\ & \bullet \end{array} = \begin{array}{c} \bullet & & \bullet \\ & \diagdown \quad \diagup \\ & \bullet \end{array} = \begin{array}{c} \bullet & & \bullet \\ & \diagdown \quad \diagup \\ & \bullet \end{array} = \begin{array}{c} \bullet & & \bullet \\ & \text{---} \\ & \bullet \end{array} = \begin{array}{c} \bullet & & \bullet \\ & \text{---} \\ & \bullet \end{array} = 0$$




Our goal is to show:

$$\begin{array}{c} \bullet \\ \bullet \end{array} > \frac{1}{6} \text{ subject to } \begin{array}{c} \bullet & & \bullet \\ & \text{red} & \\ \bullet & & \bullet \end{array} = \begin{array}{c} \bullet & & \bullet \\ & \text{blue} & \\ \bullet & & \bullet \end{array} = \begin{array}{c} \bullet & & \bullet \\ & \text{red} & \\ & & \bullet \end{array} = \begin{array}{c} \bullet & & \bullet \\ & \text{red} & \\ & & \bullet \end{array} = \begin{array}{c} \bullet & & \bullet \\ & \text{blue} & \\ & & \bullet \end{array} = 0$$

Observe that $\begin{array}{c} \bullet \\ \text{red} \\ \bullet \end{array}$ and $\begin{array}{c} \bullet \\ \text{blue} \\ \bullet \end{array}$ can be swapped.




Our goal is to show:

$$\begin{array}{c} \bullet \\ \bullet \end{array} > \frac{1}{6} \text{ subject to } \begin{array}{c} \bullet \\ \text{---} \text{---} \text{---} \bullet \\ \backslash \quad / \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \text{---} \text{---} \text{---} \bullet \\ / \quad \backslash \\ \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \backslash \quad / \\ \bullet \end{array} = \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \end{array} = 0$$

Observe that  and  can be swapped. Change to a color-blind setting.  is a monochromatic triangle (red or blue).

Our goal is to show:

$$\begin{array}{c} \bullet \\ \bullet \end{array} > \frac{1}{6} \text{ subject to } \begin{array}{c} \bullet \\ \text{---} \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \text{---} \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \text{---} \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \end{array} = 0$$




Observe that  and  can be swapped. Change to a color-blind setting.  is a monochromatic triangle (red or blue).

Our new goal is to show:

$$\begin{array}{c} \bullet \\ \bullet \end{array} > \frac{1}{6} \text{ subject to } \begin{array}{c} \bullet \\ \text{---} \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \text{---} \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \end{array} = 0$$

Our goal is to show:

$$\begin{array}{c} \bullet \\ \bullet \end{array} > \frac{1}{6} \text{ subject to } \begin{array}{c} \bullet \\ \text{---} \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \text{---} \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \text{---} \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \end{array} = 0$$

Observe that  and  can be swapped. Change to a color-blind setting.  is a monochromatic triangle (red or blue).

Our new goal is to show:

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Color-blind setting will allow us to fit the computation on these slides.

Also important for bigger applications.

Our goal is to show:

$$\begin{array}{c} \bullet \\ \bullet \end{array} > \frac{1}{6} \text{ subject to } \begin{array}{c} \bullet & \bullet \\ & \backslash / \\ & \bullet \end{array} = \begin{array}{c} \bullet & \bullet \\ & \backslash / \\ & \bullet \end{array} = \begin{array}{c} \bullet & \bullet \\ & \bullet \end{array} = 0$$

Our goal is to show:

$$\begin{array}{c} \bullet \\ \bullet \end{array} > \frac{1}{6} \text{ subject to } \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \text{---} \quad \text{---} \\ \bullet \quad \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} = 0$$

Basic equations:

$$\begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \text{---} \quad \text{---} \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \bullet \quad \bullet \end{array} = 1$$

$$\begin{array}{c} \bullet \\ \bullet \end{array} = \frac{1}{6} \left(1 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + 0 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + 0 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + 1 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + 3 \begin{array}{c} \bullet \quad \bullet \\ \text{---} \quad \text{---} \\ \bullet \quad \bullet \end{array} + 2 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + 6 \begin{array}{c} \bullet \quad \bullet \\ \bullet \quad \bullet \end{array} \right)$$

We use flags with type σ_1 of size two

$$F = \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \end{array} \right)^T.$$

For a positive semidefinite matrix M

$$\begin{aligned} 0 \leq \left[\left[F^T M F \right]_{\sigma_1} \right] &= \left[\left[F^T \begin{pmatrix} 0.0744 & -0.0223 & -0.0520 \\ -0.0223 & 0.0238 & -0.0014 \\ -0.0520 & -0.0014 & 0.0536 \end{pmatrix} F \right]_{\sigma_1} \right] \\ &= -0.0116 \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} - 0.3568 \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} - 0.1784 \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} - 0.0112 \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} \\ &\quad + 0.3216 \begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \end{array} + 0 \begin{array}{c} \text{Diagram 11} \\ \text{Diagram 12} \end{array} + 0 \begin{array}{c} \text{Diagram 13} \\ \text{Diagram 14} \end{array}. \end{aligned}$$

$$\left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \times \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right]_{\sigma_1} = \left[\frac{1}{2} \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} + \frac{1}{2} \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} \right]_{\sigma_1} = \frac{1}{2} \left(\frac{8}{12} \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} + \frac{4}{12} \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} \right).$$

$$\begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} = \frac{1}{6} \left(1 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + 0 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} + 0 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagup \\ \bullet \quad \bullet \end{array} + 1 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagdown \\ \bullet \quad \bullet \end{array} + 3 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagup \\ \bullet \quad \bullet \end{array} + 2 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagdown \\ \bullet \quad \bullet \end{array} + 6 \begin{array}{c} \bullet \quad \bullet \\ \vdots \quad \vdots \\ \bullet \quad \bullet \end{array} \right)$$

$$\begin{aligned} 0 \geq & 0.0116 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + 0.3568 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} + 0.1784 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagup \\ \bullet \quad \bullet \end{array} + 0.0112 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagdown \\ \bullet \quad \bullet \end{array} \\ & - 0.3216 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagup \\ \bullet \quad \bullet \end{array} + 0 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagdown \\ \bullet \quad \bullet \end{array} + 0 \begin{array}{c} \bullet \quad \bullet \\ \vdots \quad \vdots \\ \bullet \quad \bullet \end{array}. \end{aligned}$$

$$\begin{array}{c} \bullet \\ \bullet \end{array} = \frac{1}{6} \left(1 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + 0 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} + 0 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagup \\ \bullet \quad \bullet \end{array} + 1 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagdown \\ \bullet \quad \bullet \end{array} + 3 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagup \\ \bullet \quad \bullet \end{array} + 2 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagdown \\ \bullet \quad \bullet \end{array} + 6 \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \right)$$

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We sum the equations and obtain

$$\begin{aligned} \begin{array}{c} \bullet \\ \bullet \end{array} & \geq 0.1782 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + 0.3568 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} + 0.1784 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagup \\ \bullet \quad \bullet \end{array} + 0.1778 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagdown \\ \bullet \quad \bullet \end{array} \\ & + 0.1784 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagup \\ \bullet \quad \bullet \end{array} + 0.33 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagdown \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \end{aligned}$$

$$\begin{array}{c} \bullet \\ \bullet \end{array} = \frac{1}{6} \left(1 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + 0 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} + 0 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagup \\ \bullet \quad \bullet \end{array} + 1 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagdown \\ \bullet \quad \bullet \end{array} + 3 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagup \\ \bullet \quad \bullet \end{array} + 2 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagdown \\ \bullet \quad \bullet \end{array} + 6 \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \right)$$

$$\begin{aligned} 0 \geq & 0.0116 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + 0.3568 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} + 0.1784 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagup \\ \bullet \quad \bullet \end{array} + 0.0112 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagdown \\ \bullet \quad \bullet \end{array} \\ & - 0.3216 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagup \\ \bullet \quad \bullet \end{array} + 0 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagdown \\ \bullet \quad \bullet \end{array} + 0 \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array}. \end{aligned}$$

We sum the equations and obtain

$$\begin{aligned} \begin{array}{c} \bullet \\ \bullet \end{array} & \geq 0.1782 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + 0.3568 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} + 0.1784 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagup \\ \bullet \quad \bullet \end{array} + 0.1778 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagdown \\ \bullet \quad \bullet \end{array} \\ & + 0.1784 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagup \\ \bullet \quad \bullet \end{array} + 0.33 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagdown \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \\ & \geq 0.17 \left(\begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagup \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagdown \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagup \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagdown \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \right) \end{aligned}$$

$$\begin{array}{c} \bullet \\ \bullet \end{array} = \frac{1}{6} \left(1 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + 0 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} + 0 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagup \\ \bullet \quad \bullet \end{array} + 1 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagdown \\ \bullet \quad \bullet \end{array} + 3 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagup \\ \bullet \quad \bullet \end{array} + 2 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagdown \\ \bullet \quad \bullet \end{array} + 6 \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \right)$$

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We sum the equations and obtain



$$\begin{aligned} \begin{array}{c} \bullet \\ \bullet \end{array} & \geq 0.1782 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + 0.3568 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} + 0.1784 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagup \\ \bullet \quad \bullet \end{array} + 0.1778 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagdown \\ \bullet \quad \bullet \end{array} \\ & + 0.1784 \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagup \\ \bullet \quad \bullet \end{array} + 0.33 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagdown \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \\ & \geq 0.17 \left(\begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagup \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagdown \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagup \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagdown \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \right) \\ & > 0.17 > \frac{1}{6}. \end{aligned}$$

We proved

$$\begin{array}{c} \bullet \\ \bullet \end{array} > \frac{1}{6} \text{ subject to } \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \text{---} \\ \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \text{---} \\ \bullet \end{array} = 0$$

We proved



$$\begin{array}{c} \bullet \\ \bullet \end{array} > \frac{1}{6} \text{ subject to } \begin{array}{c} \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \bullet \end{array} = 0$$


Hence Ramsey graph for  and  has less than 6 vertices.
And therefore $R(K_3, K_3) \leq 6$.

We proved

$$\begin{array}{c} \bullet \\ \bullet \end{array} > \frac{1}{6} \text{ subject to } \begin{array}{c} \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \bullet \end{array} = 0$$



The diagrammatic equation shows a sequence of equalities between graph configurations. It starts with a 2-vertex configuration (two isolated vertices) with a coefficient greater than 1/6. This is followed by an equals sign and a sum of five 3-vertex configurations, each with a coefficient of 1/6. The configurations are: a red triangle, a blue triangle, a V-shape with one red and one blue edge, a red edge with an isolated vertex, and a blue edge with an isolated vertex. The sequence ends with an equals sign and a 0.


Hence Ramsey graph for  and  has less than 6 vertices.
And therefore $R(K_3, K_3) \leq 6$.

Note that the matrix M was not unique or tight (easy rounding).
(bound  $\geq \frac{1}{5}$ is also obtainable)

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$$\begin{array}{c} \bullet \\ \bullet \end{array} > \frac{1}{6} \text{ subject to } \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagdown \\ \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \text{---} \\ \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \text{---} \\ \bullet \end{array} = 0$$

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Thank you for your attention!