Small Ramsey Numbers

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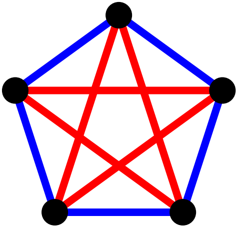
http://www.d.umn.edu/~dfroncek/MMMM_2017/MMMM_2017.pdf

Definition

$R(G_1, G_2, \ldots, G_k)$ is the smallest integer n such that any k-edge coloring of K_n contains a copy of G_i in color i for some $1 \leq i \leq k$.

$R(K_3, K_3) > 5$



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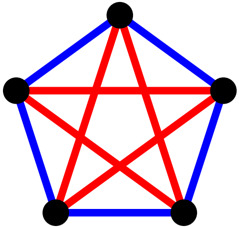

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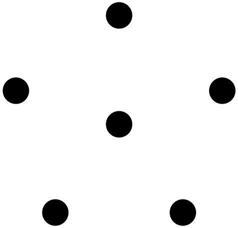

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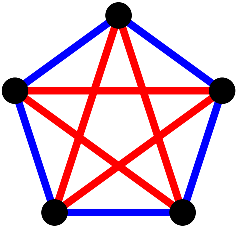

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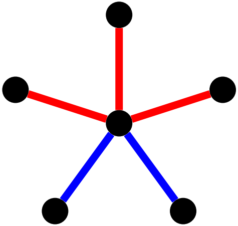

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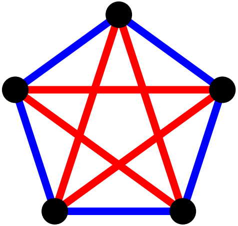

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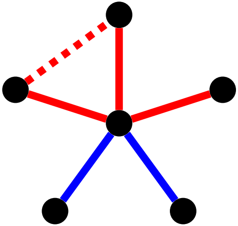

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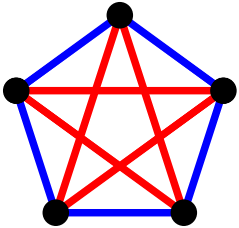

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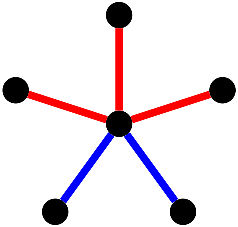

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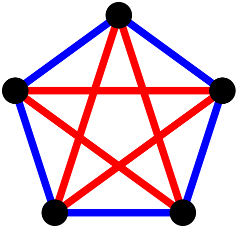

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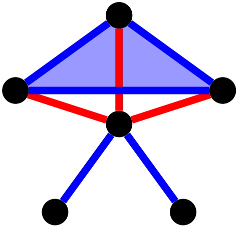

Theorem (Ramsey 1930)

$R(K_m, K_n)$ is finite.

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Questions:

- study how $R(G_1, \ldots, G_k)$ grows if $G_1, \ldots, G_k$ grow (large)
- study $R(G_1, \ldots, G_k)$ for fixed $G_1, \ldots, G_k$ (small)

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Radziszowski - *Small Ramsey Numbers*
Electronic Journal of Combinatorics - Survey

Flag algebras

Seminal paper:
Razborov, Flag Algebras, *Journal of Symbolic Logic* **72** (2007), 1239–1282.
David P. Robbins Prize by AMS for Razborov in 2013

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Example (Goodman, Razborov)

If the density of edges is at least $\rho > 0$, what is the minimum density of triangles?

- designed to attack extremal problems.
- works well if constraints as well as desired value can be computed by checking small subgraphs (or average over small subgraphs)
- the results are in limit (very large graphs)

Applications (incomplete list)

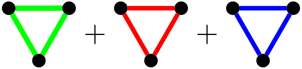
Authors	Year	Application/Result
Razborov	2008	edge density vs. triangle density
Hladký, Kráľ, Norin	2009	Bounds for the Caccetta-Haggvist conjecture
Razborov	2010	3-hypergraphs with forbidden 4-vertex configurations
Hatami, Hladký,Kráľ,Norine,Razborov / Grzesik	2011	Erdős Pentagon problem
Hatami, Hladký, Kráľ, Norin, Razborov	2012	Non-Three-Colourable Common Graphs Exist
Balogh, Hu, Lidický, Liu / Baber	2012	4-cycles in hypercubes
Reiher	2012	edge density vs. clique density
Das, Huang, Ma, Naves, Sudakov	2013	minimum number of k-cliques
Baber, Talbot	2013	A Solution to the 2/3 Conjecture
Falgas-Ravry, Vaughan	2013	Turán density of many 3-graphs
Cummings, Kráľ, Pfender, Sperfeld, Treglown, Young	2013	Monochromatic triangles in 3-edge colored graphs
Kramer, Martin, Young	2013	Boolean lattice
Balogh, Hu, Lidický, Pikhurko, Udvari, Volec	2013	Monotone permutations
Norin, Zwols	2013	New bound on Zarankiewicz's conjecture
Huang, Linial, Naves, Peled, Sudakov	2014	3-local profiles of graphs
Balogh, Hu, Lidický, Pfender, Volec, Young	2014	Rainbow triangles in 3-edge colored graphs
Balogh, Hu, Lidický, Pfender	2014	Induced density of C_5
Goaoc, Hubard, de Verclos, Séréni, Volec	2014	Order type and density of convex subsets
Coregliano, Razborov	2015	Tournaments

Applications to graphs, oriented graphs, hypergraphs, hypercubes, permutations, crossing number of graphs, order types, geometry, ...Razborov: Flag Algebra: an Interim Report

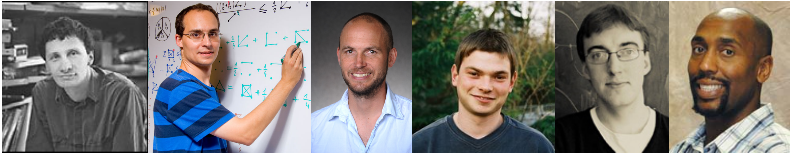
Inspiration

Theorem (Cummings, Kráľ, Pfender, Sperfeld, Treglown, Young)

In every 3-edge-colored complete graph on n vertices, there are at least $\frac{1}{25}\binom{n}{3} + o(n^3)$ *monochromatic triangles.*



$$\geq \frac{1}{25} + o(1)$$



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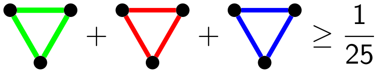

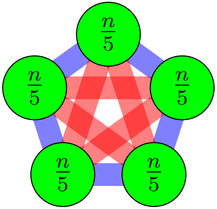

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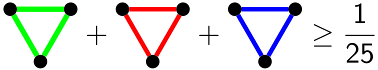

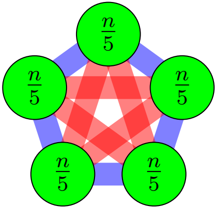

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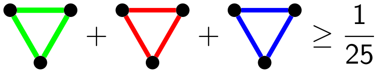

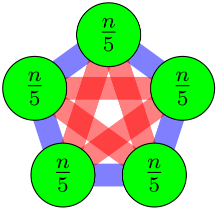

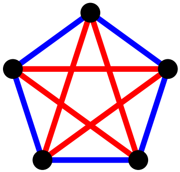

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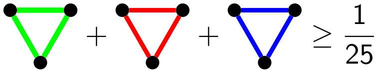

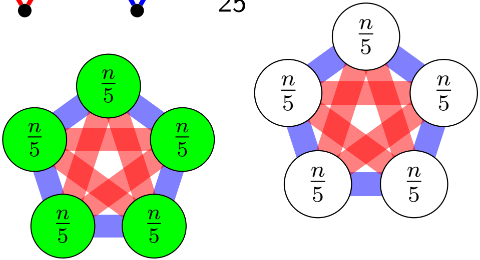

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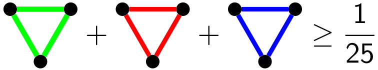

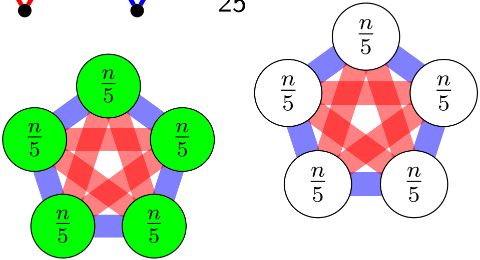

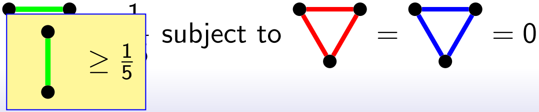

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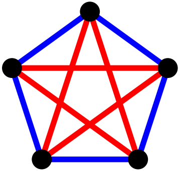

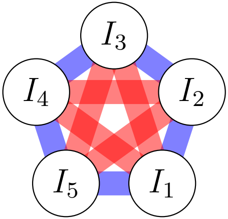

What is number of non-edges in a blow-up?

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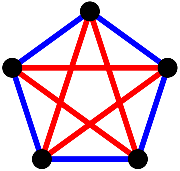

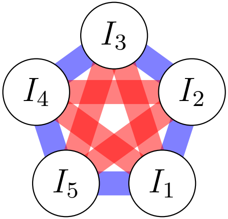

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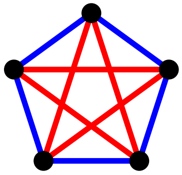

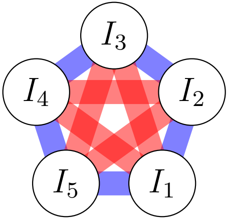

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Observation (Key observation)

If a Ramsey graph G has k vertices, then the density of non-edges in any blow-up of G is at least $\frac{1}{k}+o(1)$.

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- Let $\mathcal{G}$ be 2-edge-colored complete graphs with no monochromatic triangle.
- Consider all blow-ups $\mathcal{B}$ of graphs in $\mathcal{G}$
- $\forall B \in \mathcal{B}$, density of non-edges in B is at least $\frac{1}{k} = \frac{1}{5}$.

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Notice that any lower bound on ρ in $(\frac{1}{k+1}, \frac{1}{k}]$ gives that any Ramsey graph has at most k vertices.

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$R(G_1, \ldots, G_n) \leq 1 + 1/\rho$

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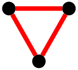
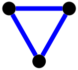
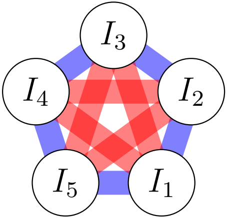
Blow-ups in Flag Algebra

How can we characterize blow-ups $\mathcal{B}$ of graphs with no  ,  ?



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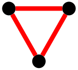
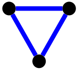
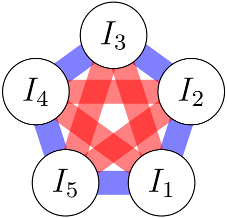
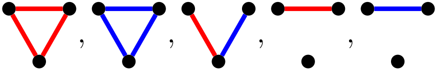
of graphs with no  ,  ?



Forbidden subgraphs :



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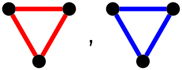
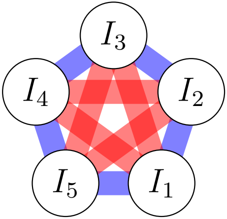
How can we characterize blow-ups $\mathcal{B}$ of graphs with no  ?



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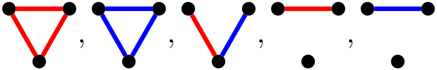
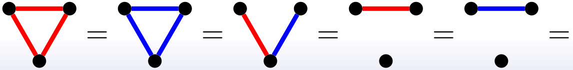


minimize 

subject to  = 0

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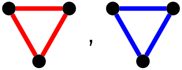
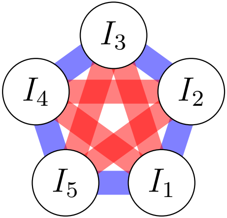
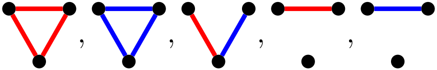
How can we characterize blow-ups $\mathcal{B}$ of graphs with no  ?

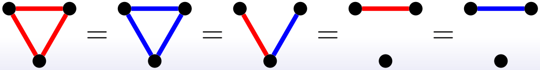


Forbidden subgraphs :



minimize 

Flag Algebra question! Easy to modify.

subject to  $= 0$

New upper bounds (so far)

Problem	Lower	New upper	Old upper
$R(K_4^-, K_8^-)$	29	32	38
$R(K_4^-, K_9^-)$	31	46	53
$R(K_4, K_7^-)$	37	49	52
$R(K_5^-, K_6^-)$	31	38	39
$R(K_5^-, K_7^-)$	40	65	66
$R(K_5, K_6^-)$	43	62	66
$R(K_5, K_7^-)$	58	102	110
$R(K_6^-, K_7^-)$	59	124	135
$R(K_7, K_4^-)$	28	29	30
$R(K_8, K_4^-)$	29	39	42
$R(K_8, C_5)$	29	29	33
$R(K_9, C_5)$	33	36	??
$R(K_9, C_6)$	41	41	??
$R(K_9, C_7)$	49	58	??
$R(K_{2,2,2}, K_{2,2,2})$	30	32	60?

Problem	Lower	New upper	Old upper
$R(K_{3,4}, K_{2,5})$		20	21
$R(K_{3,4}, K_{3,3})$		20	25
$R(K_{3,4}, K_{3,4})$		25	30
$R(K_{3,5}, K_{1,6})$	17	17	
$R(K_{3,5}, K_{2,4})$	16	20	
$R(K_{3,5}, K_{2,5})$	21	23	
$R(K_{3,5}, K_{3,3})$		24	28
$R(K_{3,5}, K_{3,4})$		29	33
$R(K_{3,5}, K_{3,5})$	30	33	38
$R(K_{4,4}, K_{4,4})$	30	49	62
$R(W_7, W_4)$		21	
$R(W_7, W_5)$		16	
$R(W_7, W_6)$		19	
$R(B_4, B_5)$	17	19	20
$R(B_3, B_6)$	17	19	22
$R(B_5, B_6)$	22	24	26

Problem	Lower	New upper	Old upper
$R(W_5, K_6)$	33	36	
$R(W_5, K_7)$	43	50	
$R(Q_3, Q_3)$	13	13	14
$R(K_3, C_5, C_5)$	17	17	21?
$R(K_3, C_4, C_4, C_4)$	24	29	??
$R(K_4, C_4, C_4)$	52	71	72
$R(K_4^-, K_4^-, K_4^-)$	28	28	30
$R(K_3, K_4^-, K_4^-)$	21	23	27
$R(K_4, K_4^-, K_4^-)$	33	47	59
$R(K_4, K_4, K_4^-)$	55	104	113
$R(K_3, K_4, K_4^-)$	30	40	41
$R(K_4^-, K_5^-; 3)$	12	12	??
$R(K_4^-, K_5; 3)$	14	16	??
$R(K_4^-, K_4^-, K_4^-; 3)$	13	14	16

$R(K_4^-, K_5^-; 3) = 12$

- No lower bound better than 10 was known.

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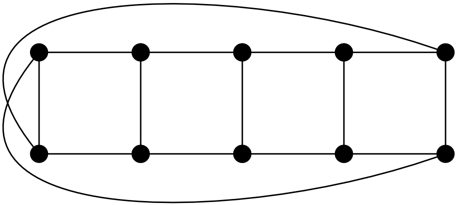
  ...and suggest that a Ramsey graph on 11 vertices can only have subgraphs on 8 vertices from a very short list.
- Generating all graphs from this short list is not difficult, and the (unique?) Ramsey graph can be found.



Example of Computation

Lemma

$R(K_3, K_3) \leq 6$

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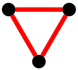
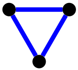
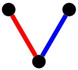
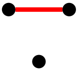
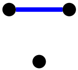
Our goal is to show:

 $> \frac{1}{6}$ subject to  $=$  $=$  $=$  $=$  $= 0$

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$R(K_3, K_3) \leq 6$

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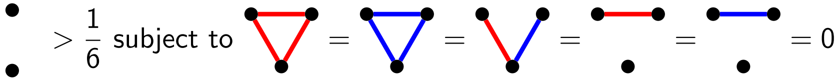
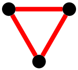
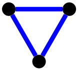
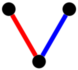
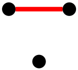
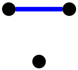


We show perhaps the most complicated proof of the lemma!

Our goal is to show:

 $> \frac{1}{6}$ subject to  $=$  $=$  $=$  $=$  $= 0$

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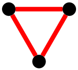
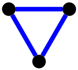
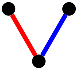
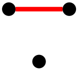
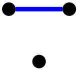
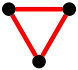
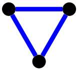
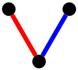
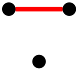
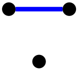
 $> \frac{1}{6}$ subject to  $=$  $=$  $=$  $=$  $= 0$

Observe that  and  can be swapped.

Our goal is to show:

 $> \frac{1}{6}$ subject to  $=$  $=$  $=$  $=$  $= 0$

Observe that  and  can be swapped. Change to a color-blind setting.  is a monochromatic triangle (red or blue).

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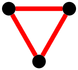
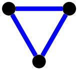
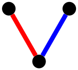
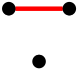
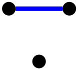
 $> \frac{1}{6}$ subject to  $=$  $=$  $=$  $=$  $= 0$

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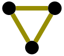
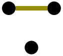
Our new goal is to show:

 $> \frac{1}{6}$ subject to  $=$  $=$  $= 0$

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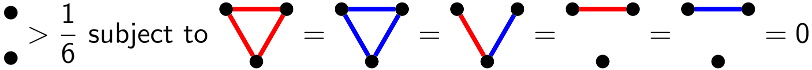

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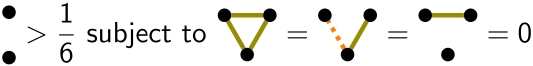
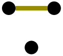


Color-blind setting will allow us to fit the computation on these slides.

Also important for bigger applications.

Our goal is to show:

 $> \frac{1}{6}$ subject to  $=$  $=$  $= 0$

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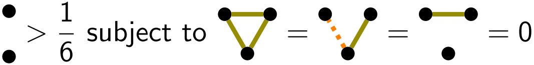
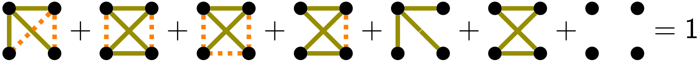
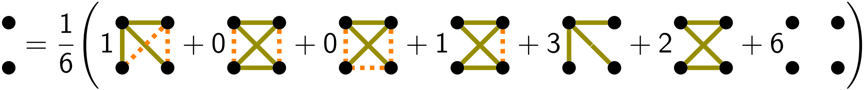
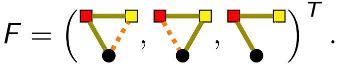


Basic equations:



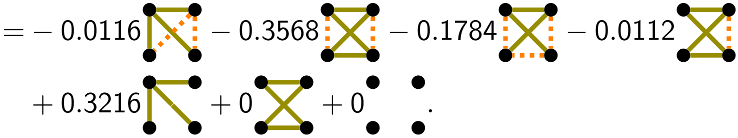


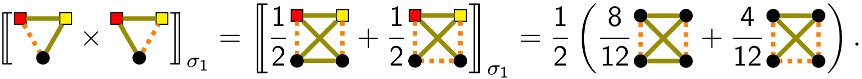
We use flags with type σ_1 of size two

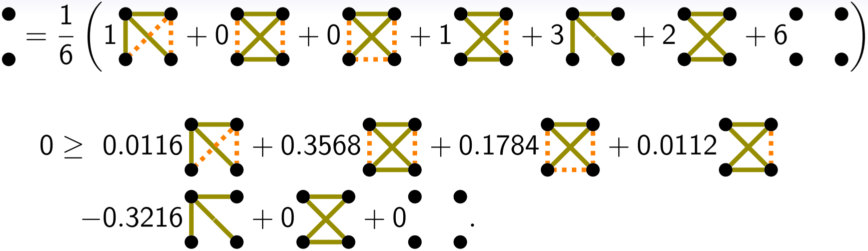


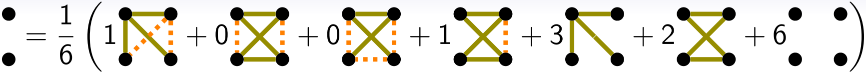
For a positive semidefinite matrix M

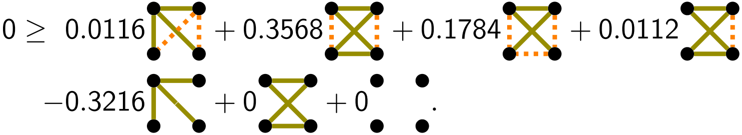
$$0 \le \left[\!\left[F^T M F \right]\!\right]_{\sigma_1} = \left[\!\left[F^T \begin{pmatrix} 0.0744 & -0.0223 & -0.0520 \\ -0.0223 & 0.0238 & -0.0014 \\ -0.0520 & -0.0014 & 0.0536 \end{pmatrix} F \right]\!\right]_{\sigma_1}$$



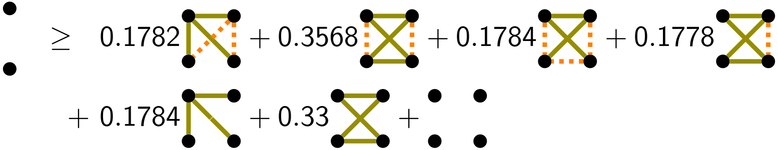


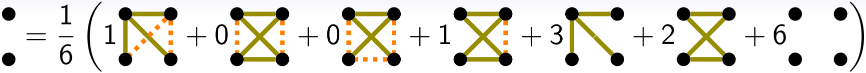

= 1/6 (1 + 0 + 0 + 1 + 3 + 2 + 6)
0 ≥ 0.0116 + 0.3568 + 0.1784 + 0.0112
−0.3216 + 0 + 0 .

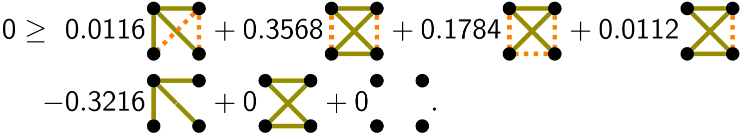




We sum the equations and obtain







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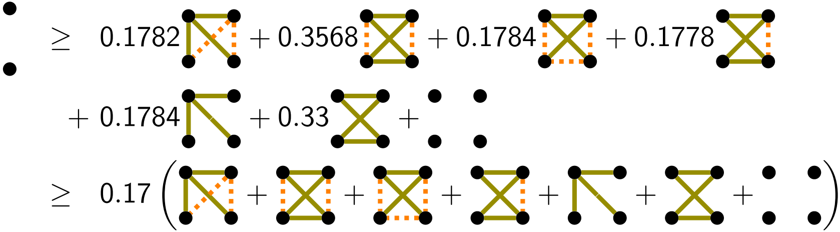

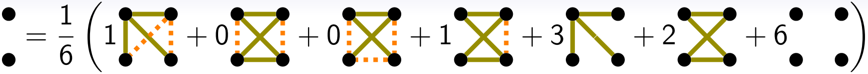

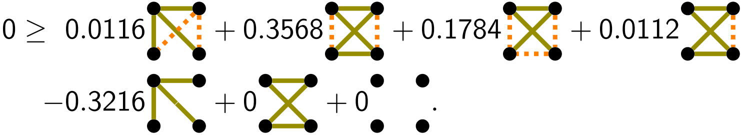

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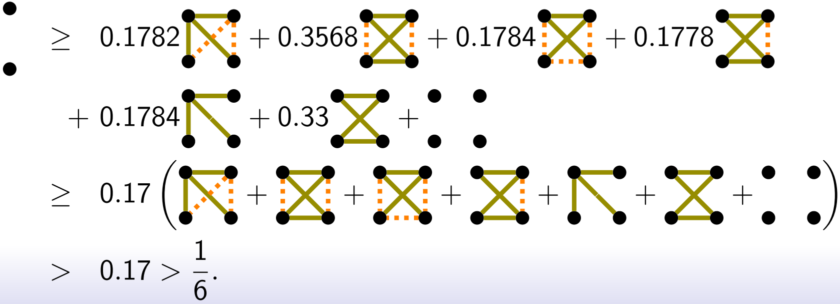
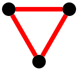
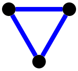
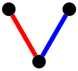
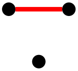
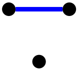
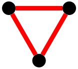
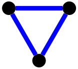
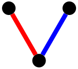
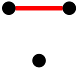
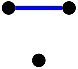
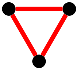
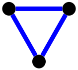


We proved

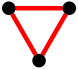
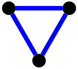
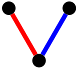
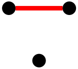
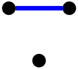
 $> \frac{1}{6}$ subject to  $=$  $=$  $=$  $=$  $= 0$

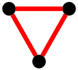
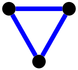
We proved

 $> \frac{1}{6}$ subject to  $=$  $=$  $=$  $=$  $= 0$

Hence Ramsey graph for  and  has less than 6 vertices.
And therefore $R(K_3, K_3) \leq 6$.

We proved

 $> \frac{1}{6}$ subject to  $=$  $=$  $=$  $=$  $= 0$

Hence Ramsey graph for  and  has less than 6 vertices.
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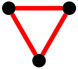
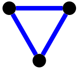
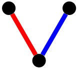
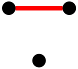
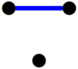
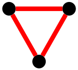
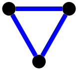
Note that the matrix M was not unique or tight (easy rounding).
(bound  $\geq \frac{1}{5}$ is also obtainable)

We proved

 $> \frac{1}{6}$ subject to  $=$  $=$  $=$  $=$  $= 0$

Hence Ramsey graph for  and  has less than 6 vertices.
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Thank you for your attention!