Decomposing graphs into edges and triangles

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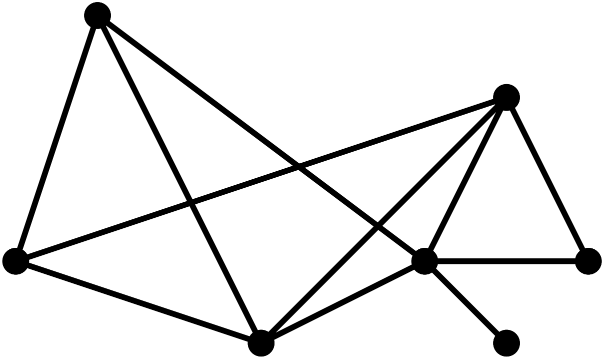
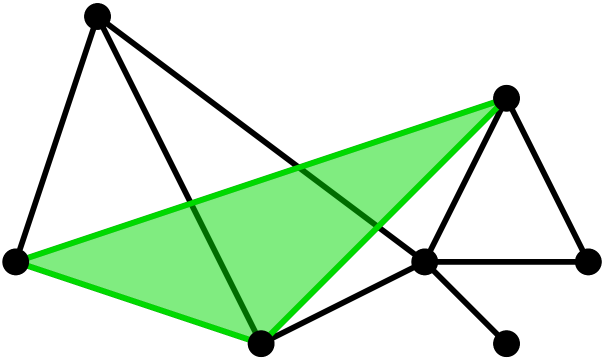
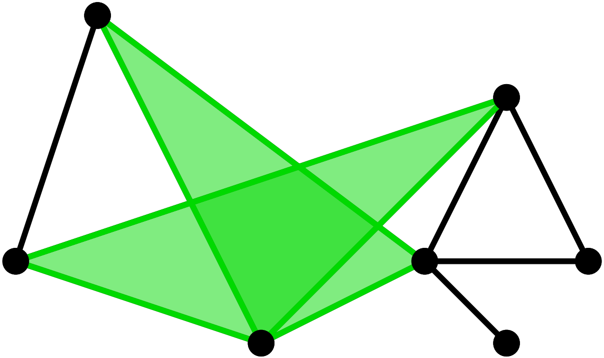
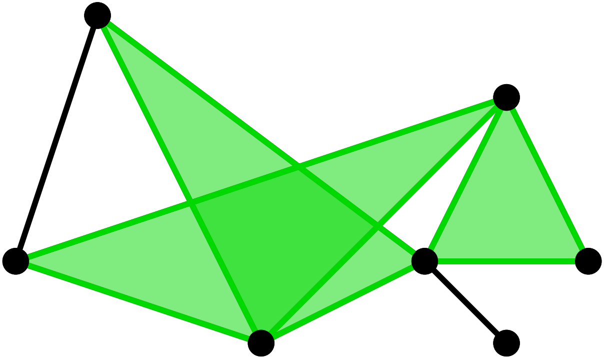
Yanitsa Pehova Taísa Martins Oleg Pikhurko
Florian Pfender Jan Volec

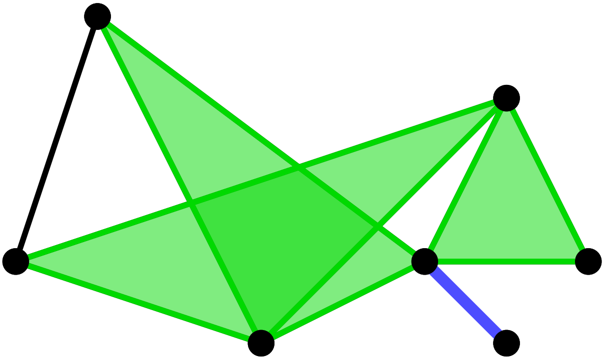
AMS Sectional Meeting #1143
October 20, 2018

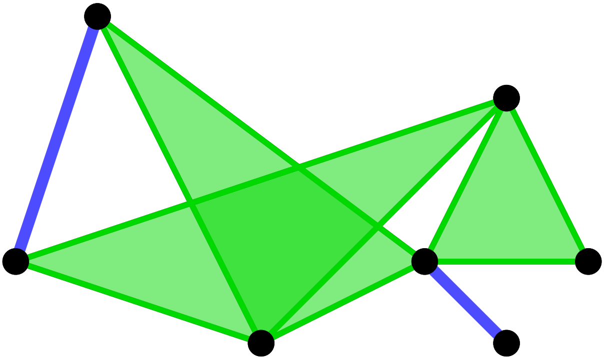












Some history

Theorem (Erdős, Goodman, Pósa (1966))

The edges of any graph G of order n can be decomposed into at most $\lfloor \frac{n^2}{4} \rfloor$ edges and triangles.

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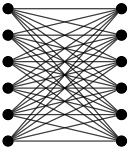
- No triangles $\rightarrow K_{\frac{n}{2},\frac{n}{2}}$ can be decomposed into $\frac{n^2}{4}$ edges;



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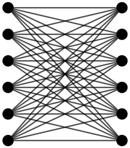
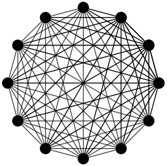
- No triangles $\rightarrow K_{\frac{n}{2},\frac{n}{2}}$ can be decomposed into $\frac{n^2}{4}$ edges;
- "All triangles" $\rightarrow K_n$ can be decomposed into $\frac{1}{3}\binom{n}{2} \approx \frac{n^2}{6}$ triangles.





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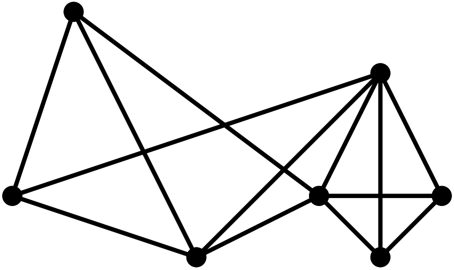
Theorem (Chung (1981); Győri, Kostochka (1980))

The edges of any graph G of order n can be decomposed into cliques $C_1, \ldots, C_\ell$ with $\sum |C_i| \leq \frac{n^2}{2}$.



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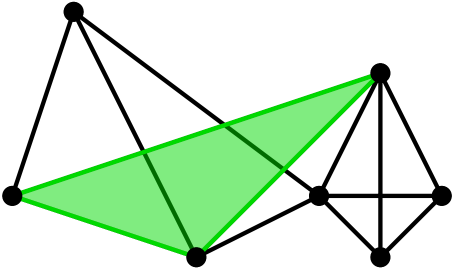

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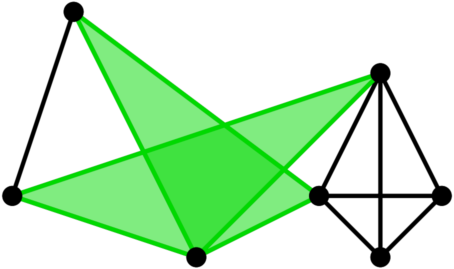

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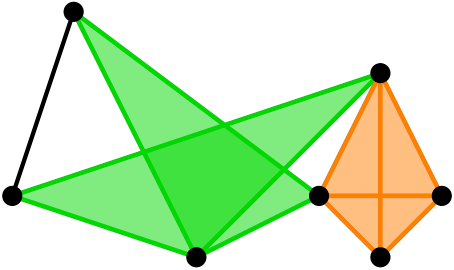

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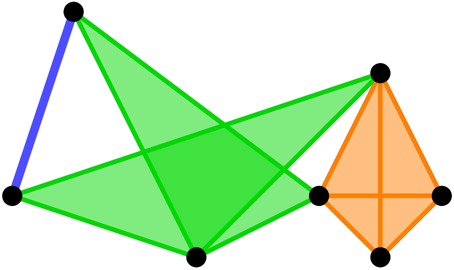

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In general:

- Assign cost c_r to a clique K_r.
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The edges of any graph G of order n can be decomposed into edges and triangles $C_1, \ldots, C_\ell$ with $\sum |C_i| \leq \frac{9n^2}{16}$.

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Conjecture (Győri and Tuza (1987))

The edges of any graph G of order n can be decomposed into edges and triangles $C_1, \ldots, C_\ell$ with $\sum |C_i| \leq \frac{n^2}{2} + o(n^2)$.

Main result 1

Theorem (Kráľ, L., Martins, Pehova 2018)

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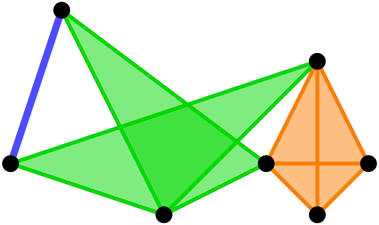

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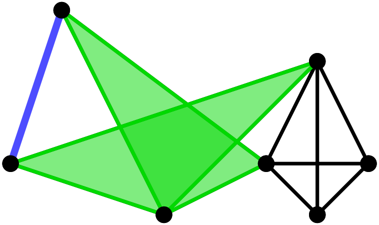

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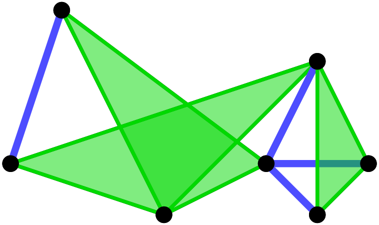

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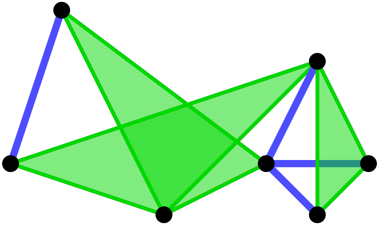


Proof outline

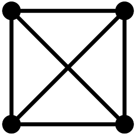
1. Obtain a fractional decomposition into edges and triangles. (flag algebras method)
2. Fractional to full decomposition. (regularity method)

Fractional decomposition

Definition

A *decomposition* of a graph G into triangles $\mathcal{T}$ and edges $\mathcal{E}$ is an assignment $w : \mathcal{T} \cup \mathcal{E} \to \{ 0, 1 \}$ such that for each $e \in E(G)$:

$$\sum_{T \supseteq e} w(T) + \sum_{e \in \mathcal{E}} w(e) = 1.$$



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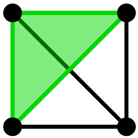

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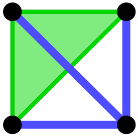

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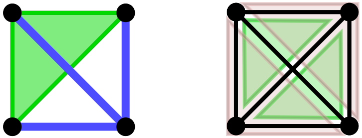

For example: K_4 doesn't have a triangle decomposition but has a fractional triangle decomposition.

Finding a fractional decomposition

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Clearly, $\pi_{3,f}(G) \leq \pi_3(G)$.

Our Theorem: $\pi_3(G) \leq \frac{n^2}{2} + o(n^2)$

Finding a fractional decomposition

Key lemma

Let G be a graph. Then

$$\mathbb{E}_W \pi_{3,f}(G[W]) \leq 21 + o(1)$$

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For every $W \in \binom{V}{7}$ take $\frac{\pi_{3,f}(G[W])}{\binom{n-2}{5}}$ decomposition and sum these.

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$$\pi_{3,f}(G) \leq \frac{1}{\binom{n-2}{5}} \sum_W \pi_{3,f}(G[W])$$
$$\leq \frac{1}{\binom{n-2}{5}} \binom{n}{7} (21 + o(1)) = \frac{n^2}{2} + o(n^2).$$

Main result

Theorem (Kráľ, L., Martins, Pehova 2018)

The edges of any graph G of order n, $\pi_3(G) \leq \frac{n^2}{2} + o(n^2)$.

Proof

✓ Obtain a fractional decomposition into edges and triangles $\pi_{3,f}(G) \leq \frac{n^2}{2} + o(n^2)$ (flag algebras method).

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- Fractional to full decomposition (regularity method).

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For any graph H and an n-vertex graph G we have

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Corollary

$$\pi_3(G) \leq \pi_{3,f}(G) + o(n^2) \leq \frac{n^2}{2} + o(n^2)$$

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Corollary

$$\pi_3(G) \leq \pi_{3,f}(G) + o(n^2) \leq \frac{n^2}{2} + o(n^2)$$

Theorem (Yuster 2004)

For a fixed family $\mathcal{F}$ of graphs and an n vertex graph G we have

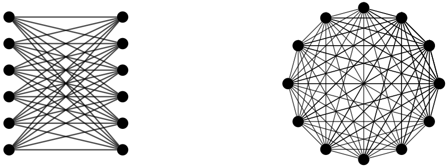
$$\nu_{\mathcal{F}}^f(G) \leq \nu_{\mathcal{F}}(G) + o(n^2).$$

Main Result 2

Theorem (Blumenthal, L., Pikhurko, Pehova, Pfender, Volec)

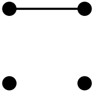
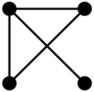
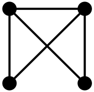
For sufficiently large n,

$$\pi_3(G) \leq \begin{cases} \frac{n^2}{2} & \textit{if} \quad n \equiv 0,2 \mod 6 \quad \ldots K_{\frac{n}{2},\frac{n}{2}} \textit{ and } K_n, \\ \frac{n^2-1}{2} & \textit{if} \quad n \equiv 1,3,5 \mod 6 \quad \ldots K_{\frac{n-1}{2},\frac{n+1}{2}}, \\ \frac{n^2}{2}+1 & \textit{if} \quad n \equiv 4 \mod 6 \quad \ldots K_n. \end{cases}$$

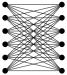
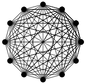


If $\pi_{3,f}(G) \leq \left(\frac{1}{2} - \varepsilon\right) n^2$ then $\pi_3(G) \leq \frac{1}{2} n^2$ by Yuster/Haxell,Rödl.

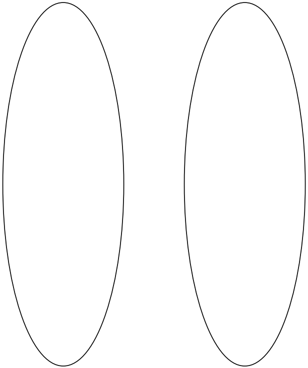
If $\pi_{3,f}(G) \geq \left(\frac{1}{2} - \varepsilon\right) n^2$, by flag algebras then the following graphs

$\mathcal{F} = \{$  ,  ,  $\}$

have density at most $\delta \geq 0$, where $\delta \to 0$ as $\varepsilon \to 0$.
By Induced removal lemma, G is $\mathcal{F}$-free up to $\delta' n^2$ edges.

Hence G is  or  up to $\delta' n^2$ edges.

Exact result $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$



- Take maxcut ($|E(G)| \geq \frac{n^2}{4}$)

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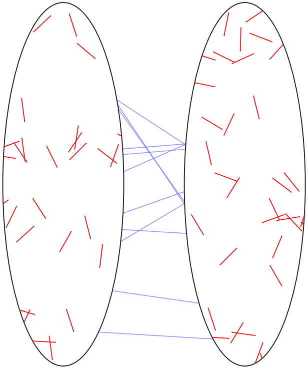

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- Bad edges

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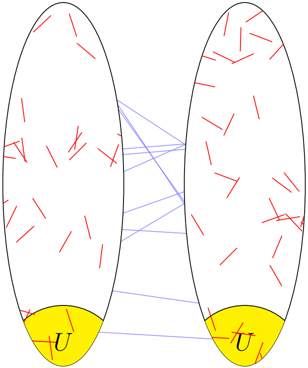

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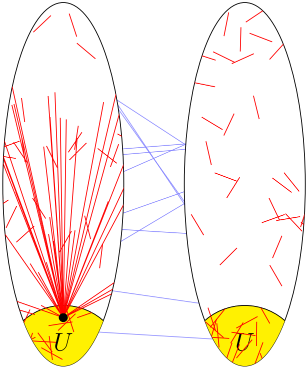

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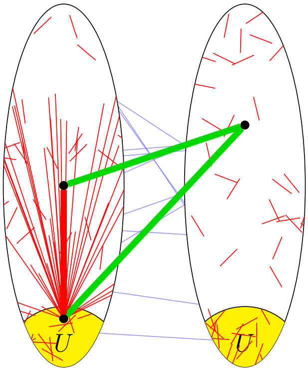

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- Triangles with 1 bad edge with U

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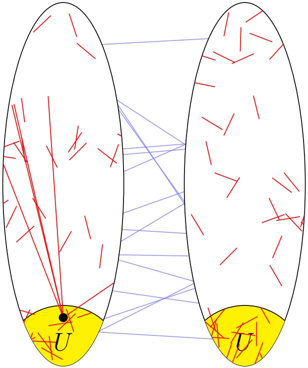

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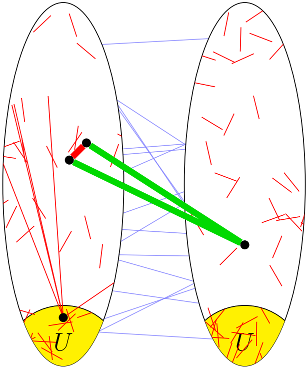
Exact result $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$



- Take maxcut ($|E(G)| \geq \frac{n^2}{4}$)
- Bad edges
- very bad vertices U
- Triangles with 1 bad edge with U
- Other triangles with bad edges

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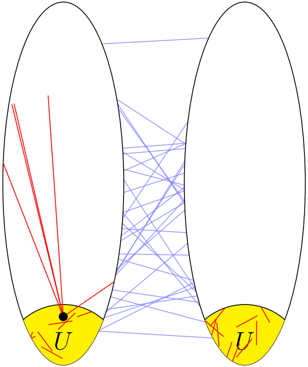

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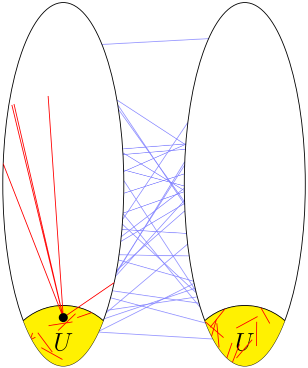

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Theorem (Győri 1988)

If G is a graph with n vertices and $n^2/4 + k$ edges, where $n \to \infty$ and $k = o(n^2)$, then it has at least $k - O(k^2/n^2)$ edge-disjoint triangles.

Exact result K_n

Theorem (Barber, Kuhn, Lo, Osthus; Dross; Gustavsson)
Every large graph G on n vertices, where $|E(G)|$ is a multiple of 3 and all vertices have even degree at least $(9/10 + o(1))n$ has a triangle decomposition.

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Every large graph G on n vertices, where $|E(G)|$ is a multiple of 3 and all vertices have even degree at least $(9/10 + o(1))n$ has a triangle decomposition.

- clean low degree vertices
- even degrees and $|E(G)|$ is a multiple of 3
- apply Theorem

One More Problem

Problem (Pyber 1991)

Can the edge set of every n-vertex graph be covered with triangles of weight 3 and edges of weight 2 such that their total weight is at most $\lfloor \frac{n^2}{2} \rfloor$?

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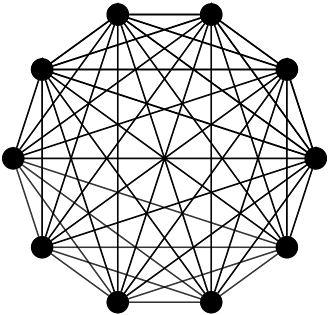
Theorem (Blumenthal, L., Pikhurko, Pehova, Pfender, Volec)

For sufficiently large n,

$$\pi_3(G) \leq \begin{cases} \frac{n^2}{2} & \text{if} \quad n \equiv 0, 2 \mod 6 \quad \ldots K_{\frac{n}{2},\frac{n}{2}} \text{ and } K_n, \\ \frac{n^2-1}{2} & \text{if} \quad n \equiv 1, 3, 5 \mod 6 \quad \ldots K_{\frac{n-1}{2},\frac{n+1}{2}}, \\ \frac{n^2}{2} + 1 & \text{if} \quad n \equiv 4 \mod 6 \quad \ldots K_n. \end{cases}$$

Pyber's problem

$n = 6k + 4$, find a covering of $G = K_n$ of cost $\leq \frac{n^2}{2}$.
Triangle decomposition: even degrees, $|E(G)|$ divisible by 3



$d(v) = 6k+3$
$e(G) = 3(6k^2 + 7k + 2)$

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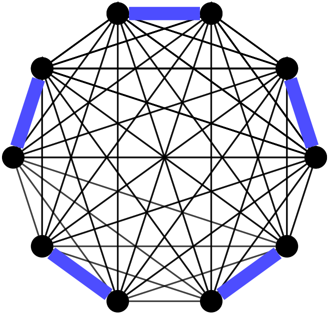

$d(v) = 6k{+}2$
$e(G) = 3(6k^2 + 7k + 2) - 3k - 2$

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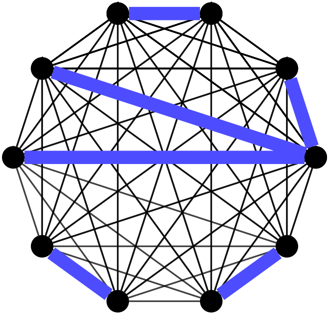

$d(v) = 6k{+}2$ or $6k$
$e(G) = 3(6k^2 + 7k + 2) - 3k - 3$
$cost = \frac{n^2}{2} + 1$

Pyber's problem

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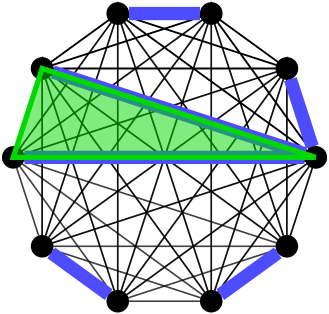

$d(v) = 6k{+}2$ or $6k$
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$cost = \frac{n^2}{2} + 1 - 4 + 3$

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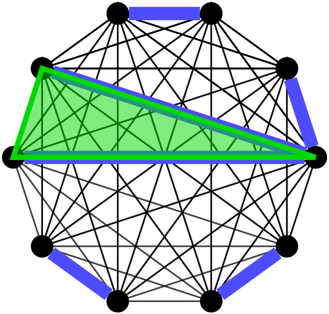

$d(v) = 6k{+}2$ or $6k$
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Thank you for your attention