Polychromatic Colorings of Complete Graphs with Respect to 1-,2-factors and Hamiltonian Cycles

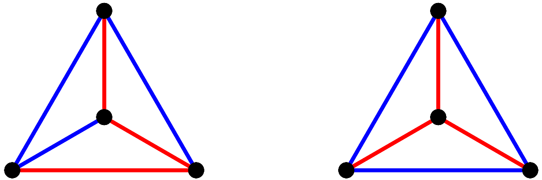
Maria Axenovich John Goldwasser Ryan Hansen
Bernard Lidický Ryan R. Martin David Offner
John Talbot Michael Young

SIAM DM
June 6, 2018

Polychromatic Coloring

Let G and H be graphs and C a set of colors.
Let $\varphi : E(G) \to C$ (not necessarily proper edge-coloring)

φ is an *H-polychromatic coloring* of G if *every* subgraph of G isomorphic to H contains *all* colors in C.



Example $H = K_3$, $G = K_4$, $C = \{red, blue\}$.

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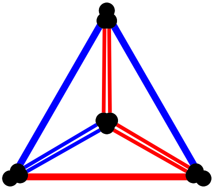

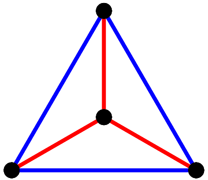

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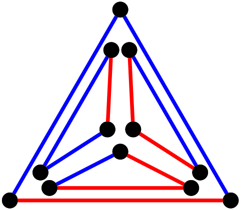

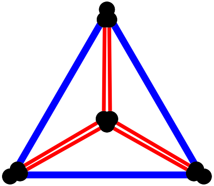

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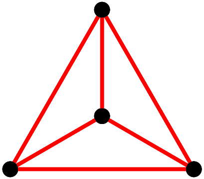

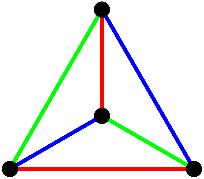

Easier to find φ with fewer colors.

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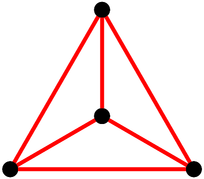

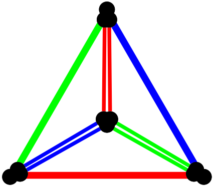

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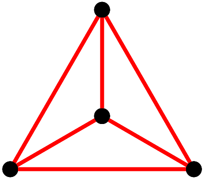

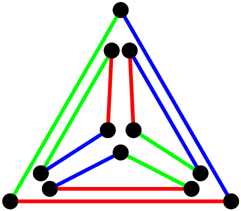

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H-polychromatic number of G is the *maximum* number of colors k such that there exists a polychromatic coloring of G with respect to H using k colors. Notation $\mathrm{poly}_H(G) = k$

Example

$$\mathrm{poly}_{K_3}(K_4) = 3$$

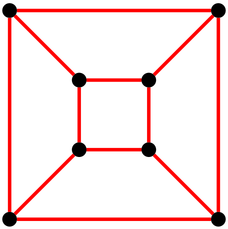
Motivation for H-polychromatic Number

Let Q_d be a d-dimensional hypercube.

Problem

What is the lergest $X \subseteq E(Q_n)$ such that $Q_n[X]$ is Q_d-free?
$ex(Q_n, Q_d)$?

Example for Q_2 in Q_3.



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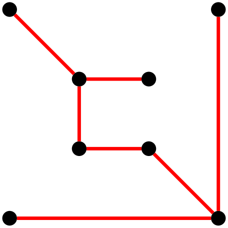

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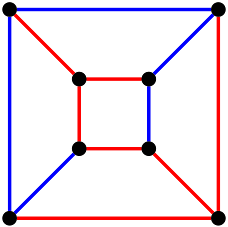

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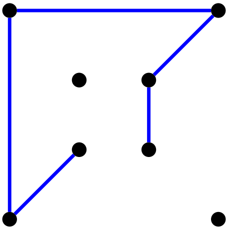

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Known Results

Theorem (Alon, Krech, Szabó 2007)

$$\binom{d+1}{2} \geq \mathrm{poly}_{Q_d}(Q_n) \geq \begin{cases} \frac{(d+1)^2}{4} & \textit{if } d \textit{ is odd} \\ \frac{d(d+2)}{4} & \textit{if } d \textit{ is even} \end{cases}$$

Theorem (Offner 2008)

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Anti-Ramsey

Edge coloring of H is *rainbow* if no two edges of H receive the same color.
Edge coloring of G is *H-anti-ramsey* if NO copy of H in G is rainbow.
$ar(G, H)$ is the largest number of colors used in an H-anti-Ramsey coloring of G.

$$ar(G, H) \leq ex(G, H)$$

$$ar(G, H) \geq \left(1 - \frac{2}{\mathrm{poly}_H(G)}\right) e(G)$$

Polychromatic Coloring of Integers

Let $S \subset \mathbb{Z}$ be finite.
Coloring of $\mathbb{Z}$ is *S-polychromatic* if every translation of S contains all colors.

Example: $S = \{0, 1, 4, 5\}$

All about this during the next talk in this session by John Goldwasser.

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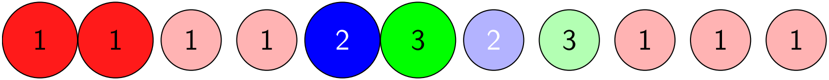

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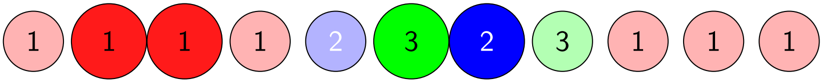

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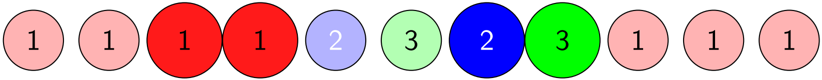

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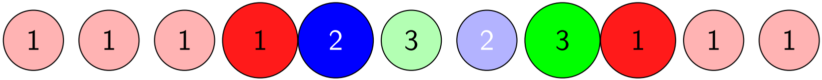

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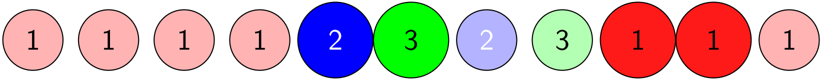

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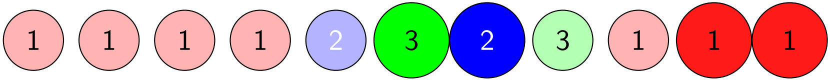

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Our Results for This Talk

Let F_k be a k-factor and HC be a Hamiltonian Cycle.

Theorem (AGHLMOTY '18)

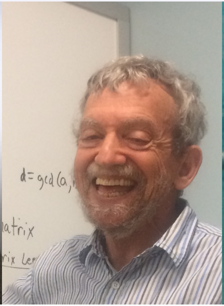
If n is an even positive integer, then $\mathrm{poly}_{F_1}(K_n) = \lfloor \log_2 n \rfloor$.

Theorem (AGHLMOTY '18)

There exists a constant c such that

$$\lfloor \log_2 2(n+1) \rfloor \leq \mathrm{poly}_{F_2}(K_n) \leq \mathrm{poly}_{HC}(K_n) \leq \log_2 n + c.$$

Exact solution for $\mathrm{poly}_{F_2}(K_n)$ and $\mathrm{poly}_{HC}(K_n)$ by G&H.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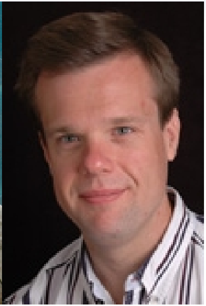
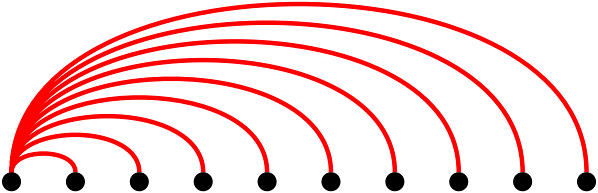


Constructions For Lower Bounds

$\lfloor \log_2 n \rfloor \leq \mathrm{poly}_{F_1}(K_n)$



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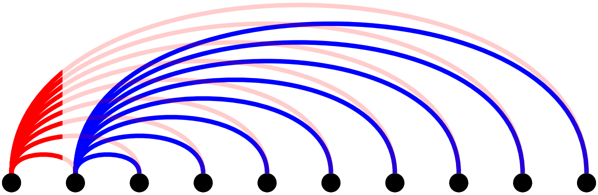

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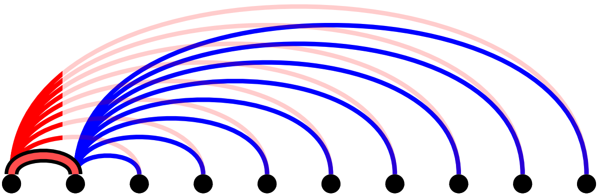

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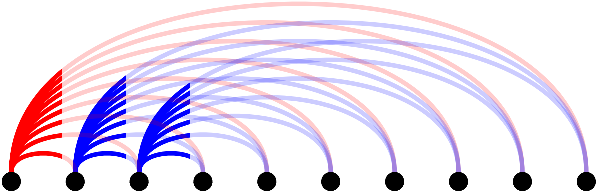

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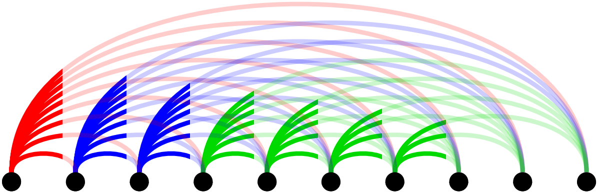

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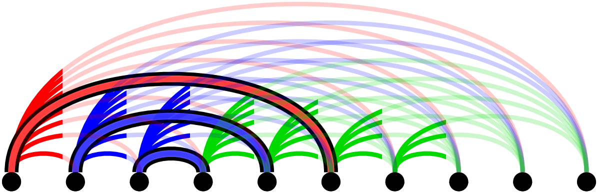

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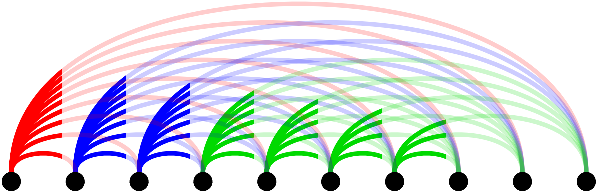

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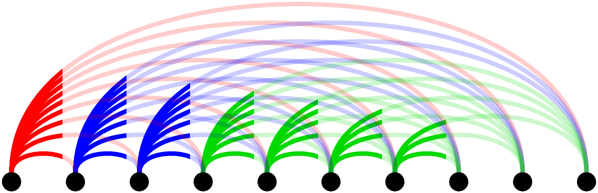

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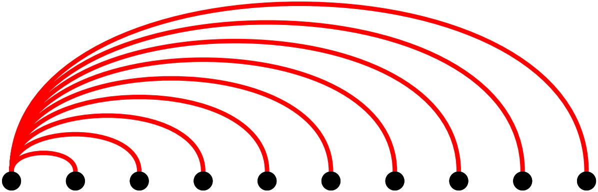

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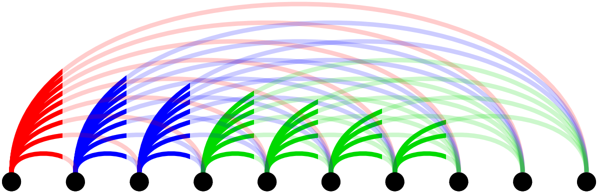

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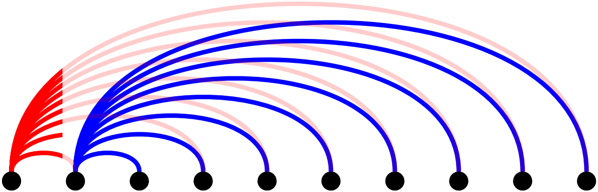

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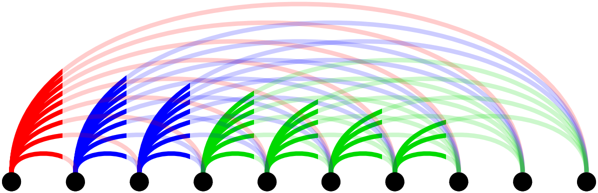

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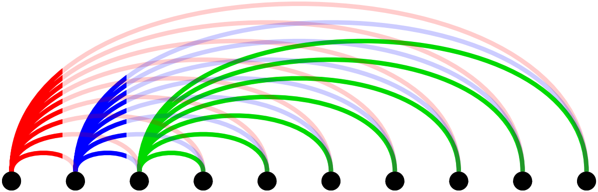

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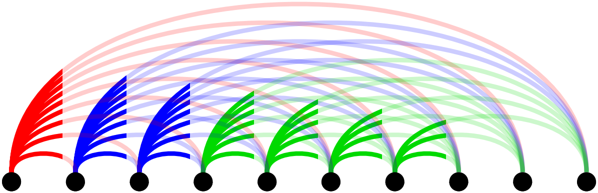

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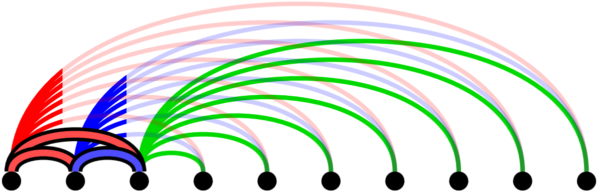

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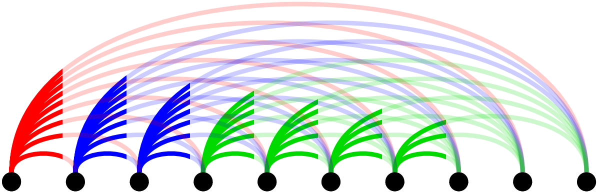

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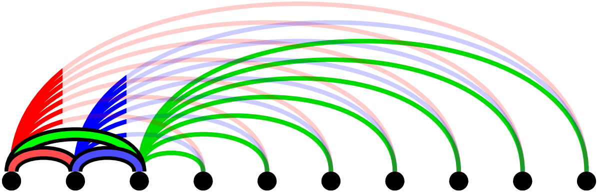

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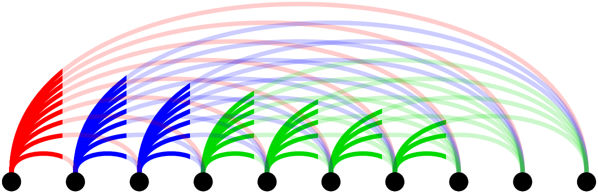

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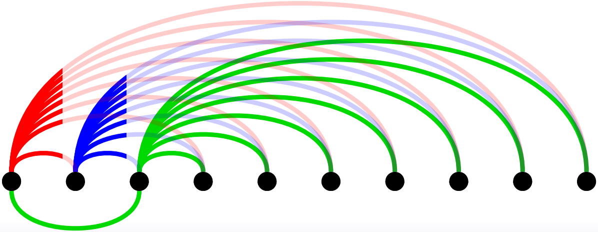

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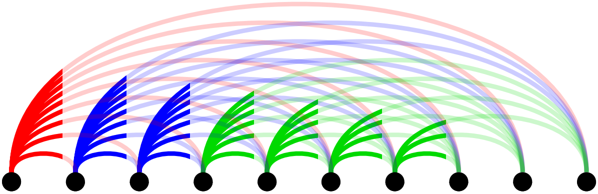

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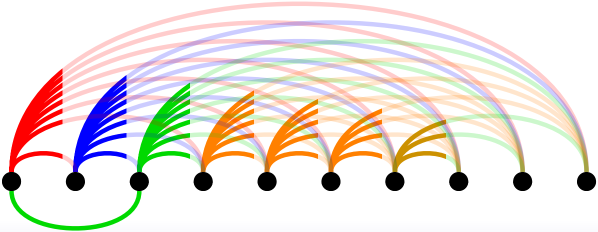

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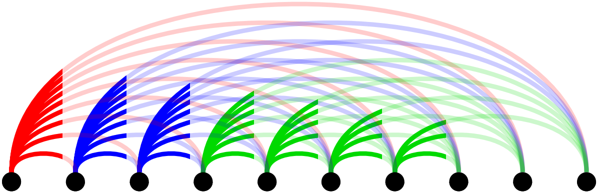

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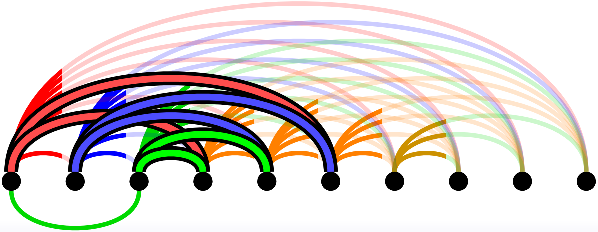


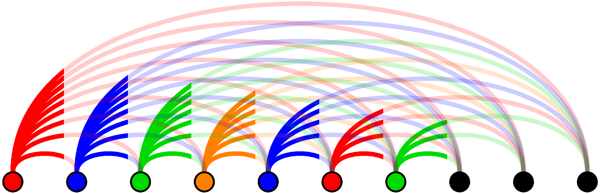
Upper bound for $\mathrm{poly}_{F_1}(K_n)$

- show there is an optimal coloring that has ordering of vertices such that for each fixed vertex v "all edges going to the right have the same color".

- for ever vertex define *inherited color*, counting argument using majority.

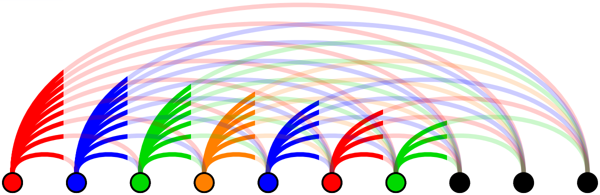
Upper bound for $\mathrm{poly}_{F_1}(K_n)$

- show there is an optimal coloring that has ordering of vertices such that for each fixed vertex v "all edges going to the right have the same color".



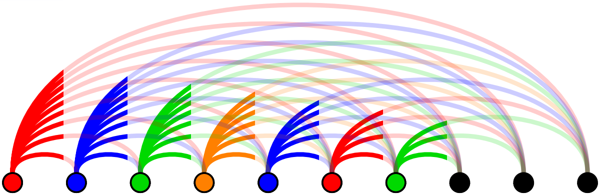
- for ever vertex define *inherited color*, counting argument using majority.

Counting first for $\mathrm{poly}_{F_1}(K_n)$



for vertex define *inherited color*

COUNTING FIRST FOR $\mathrm{poly}_{F_1}(K_n)$

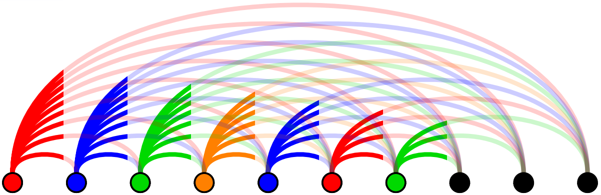


for vertex define *inherited color*

Let M_c be vertices colored color $c \in \{1, 2, \ldots\}$.

Feature: $\forall c$ exists $i_c \in [n]$ such that $|M_c \cap \{v_1, \ldots, v_i\}| > i/2$.

Counting first for $\mathrm{poly}_{F_1}(K_n)$



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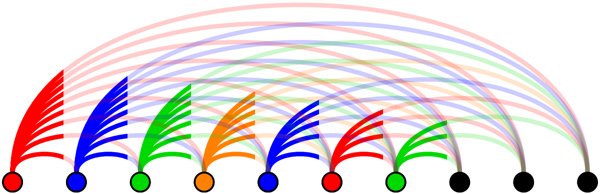

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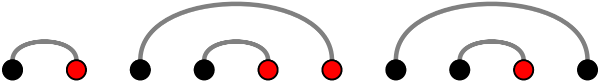
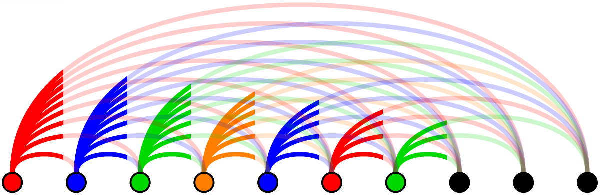


COUNTING FIRST FOR $\mathrm{poly}_{F_1}(K_n)$



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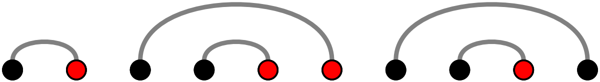
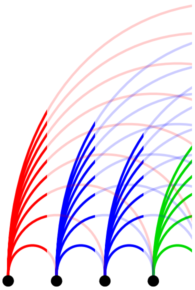


Assume that $i_1 < i_2 < \ldots$. By induction $|M_c| \geq 2^c - 1$.

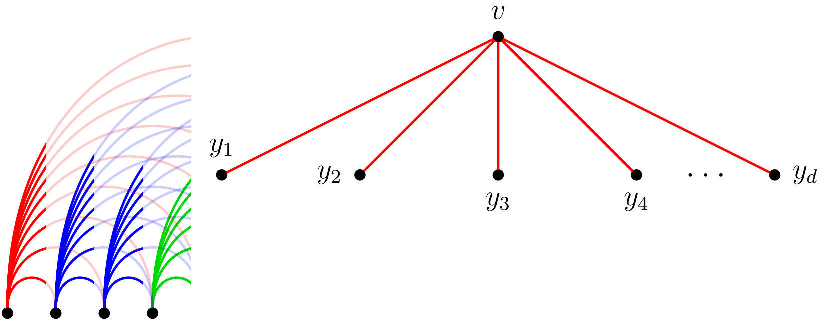
$$\sum_c |M_c| \leq n \implies c \leq \lfloor \log_2 n \rfloor$$

Ordering the vertices



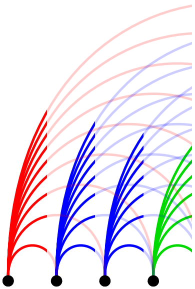
Take largest ordered initial segment,

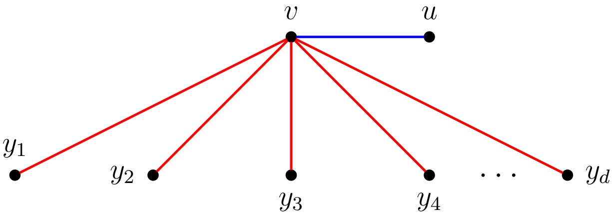
Ordering the vertices



Take largest ordered initial segment, v has maximum monochromatic degree (red) in the rest,

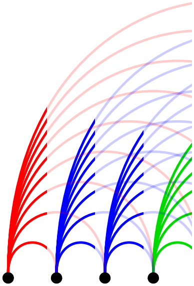
Ordering the vertices

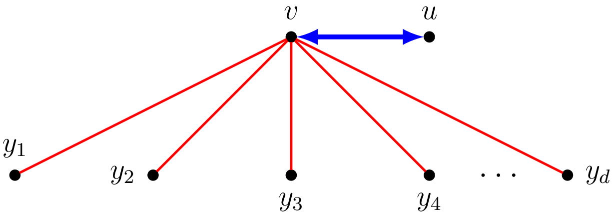




Take largest ordered initial segment, v has maximum monochromatic degree (red) in the rest, exists not red uv,

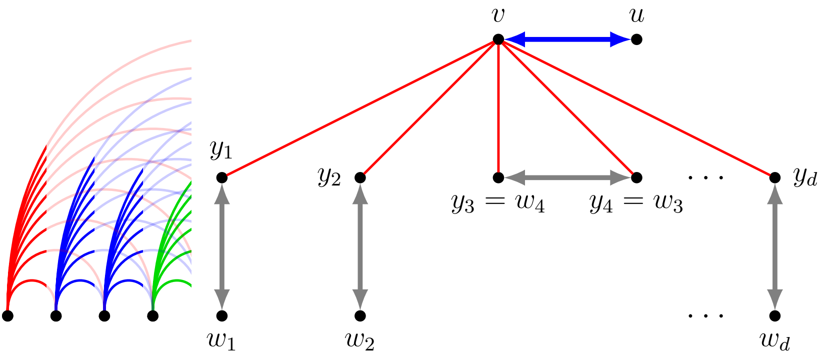
Ordering the vertices



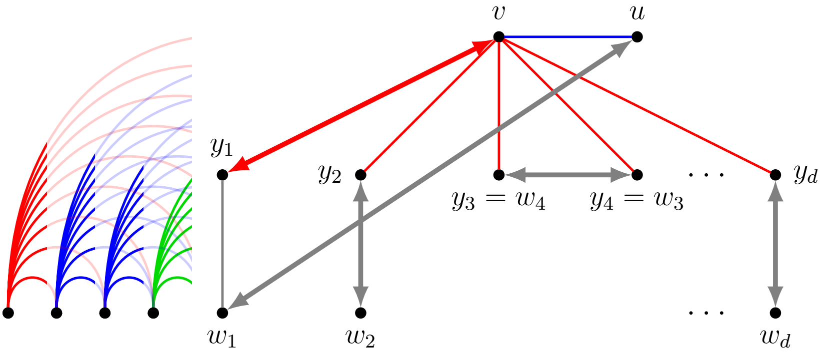


Take largest ordered initial segment, v has maximum monochromatic degree (red) in the rest, exists not red uv, $y_i w_i$ cannot be blue,

Ordering the vertices



Take largest ordered initial segment, v has maximum monochromatic degree (red) in the rest, exists not red uv, $y_i w_i$ cannot be blue, all uw_i are blue and w_i is not in the ordered segment,



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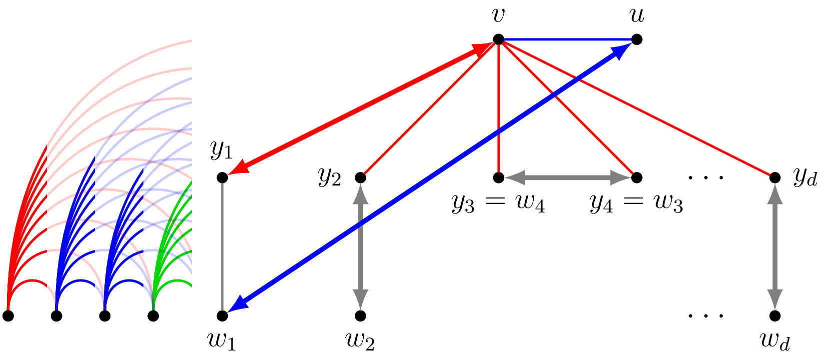
Ordering the vertices



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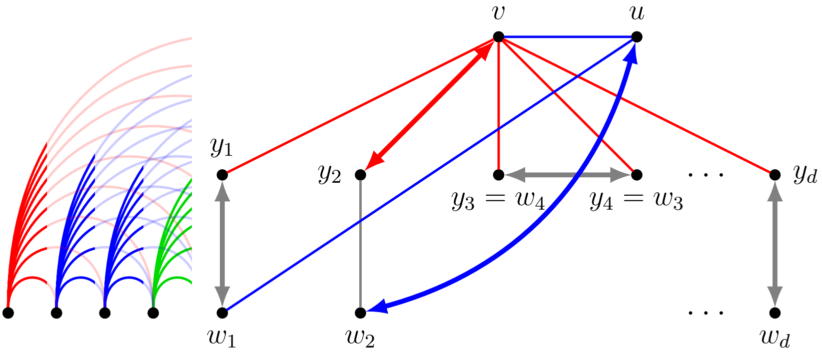
Ordering the vertices



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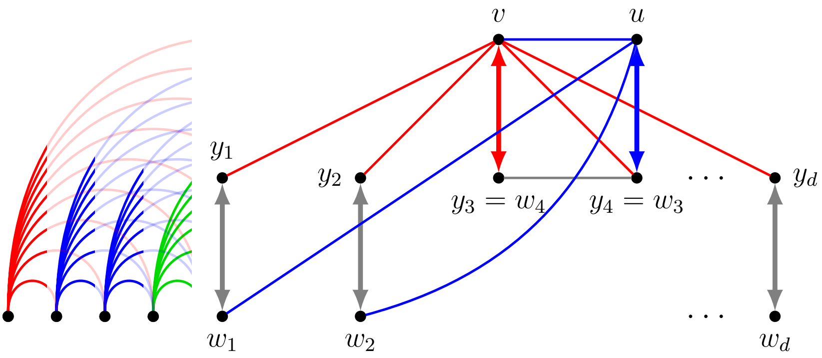
Ordering the vertices



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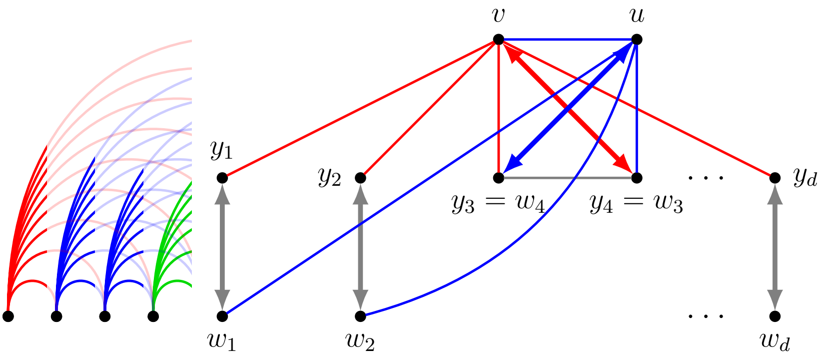
Ordering the vertices



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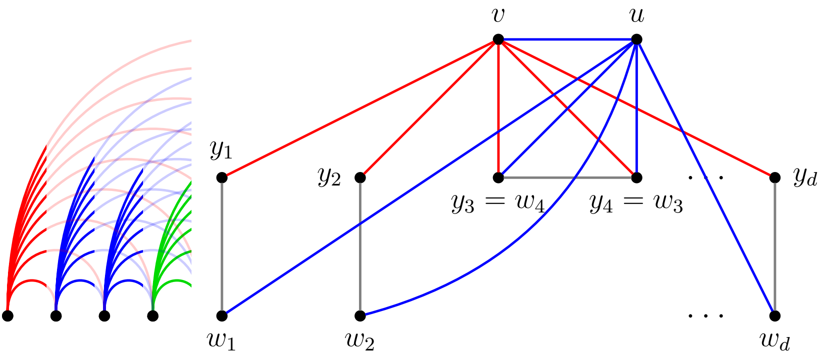
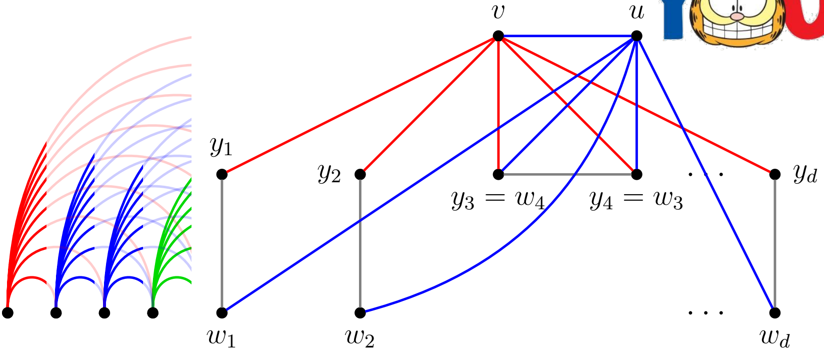
Ordering the vertices



Take largest ordered initial segment, v has maximum monochromatic degree (red) in the rest, exists not red uv, $y_i w_i$ cannot be blue, all uw_i are blue and w_i is not in the ordered segment,
u has higher mono degree than v. □

Ordering the vertices



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Thank you