Maximum Number of Almost Similar Triangles in the Plane

József Balogh Bernard Lidický Felix Christian Clemen

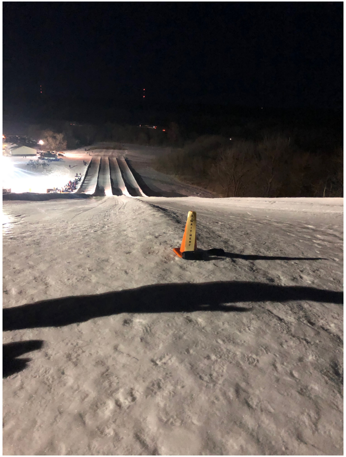
Georgia Tech Graph Theory Seminar

April 26, 2021


Calgary
Vancouver
Regina
Winnipeg
Quebec City
MAINE
Montreal
Ottawa
VERMONT
NEW HAMPSHIRE
MASSACHUSETTS
Seattle
WASHINGTON
Portland
NORTH DAKOTA
MONTANA
MINNESOTA
Toronto
NEW YORK
RI
CT
Minneapolis
WISCONSIN
MICHIGAN
New York
OREGON
SOUTH DAKOTA
Detroit
PENNSYLVANIA
NJ
IDAHO
Chicago
WYOMING
IOWA
OHIO
DE
Washington
NEBRASKA
INDIANA
ILLINOIS
Indianapolis
VIRGINIA
NEVADA
United States
Kansas City
San Francisco
UTAH
COLORADO
KANSAS
MISSOURI
KENTUCKY
NORTH CAROLINA
San Jose
Nashville
Charlotte
CALIFORNIA
TENNESSEE
SOUTH CAROLINA
Las Vegas
Atlanta
OKLAHOMA
ARKANSAS
Los Angeles
MISSISSIPPI
ALABAMA
GEORGIA
ARIZONA
NEW MEXICO
Jacksonville
San Diego
Dallas
Phoenix
Orlando
Tucson
El Paso
TEXAS
LOUISIANA
BAJA CALIFORNIA
New Orleans
Austin
Houston
FLORIDA
SONORA
Miami
The Bahamas
CHIHUAHUA
San Antonio
COAHUILA

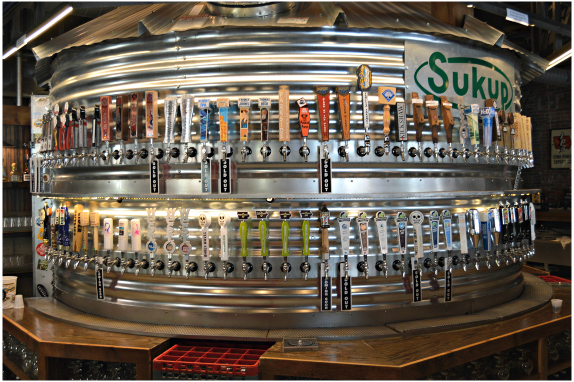
- ISU Math building in the center.


- Beach and skiing in Ames

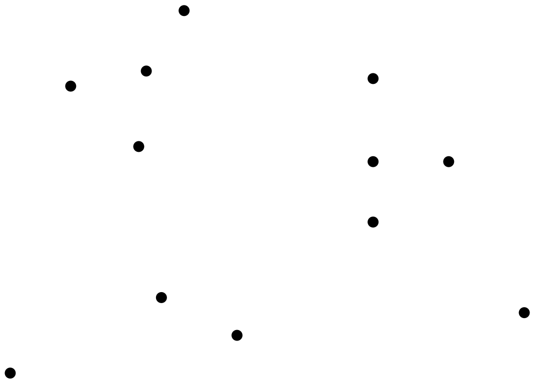
- Beach and skiing in Ames or snowtubing

- Wine



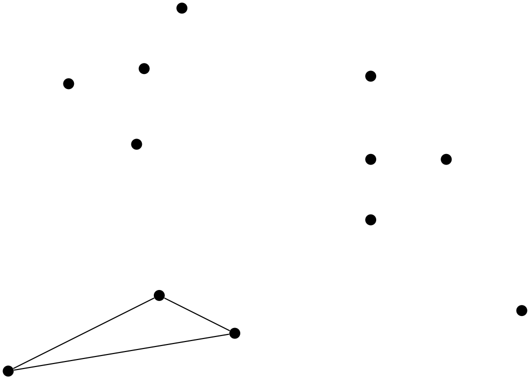
- Wine and beer

Point set in the plane



n points in the plane $\mathbb{R}^2$

Point set in the plane



n points in the plane $\mathbb{R}^2$

Definition

Let T, T' be triangles with angles $\alpha \leq \beta \leq \gamma$ and $\alpha' \leq \beta' \leq \gamma'$ respectively. The triangle T' is *ε-similar* to T if $|\alpha - \alpha'| < \varepsilon, |\beta - \beta'| < \varepsilon,$ and $|\gamma - \gamma'| < \varepsilon$.

Definition

Let T, T' be triangles with angles $\alpha \leq \beta \leq \gamma$ and $\alpha' \leq \beta' \leq \gamma'$ respectively. The triangle T' is ε-*similar* to T if $|\alpha - \alpha'| < \varepsilon, |\beta - \beta'| < \varepsilon,$ and $|\gamma - \gamma'| < \varepsilon$.

Definition

$h(n, T, \varepsilon) =$ The maximum number of triangles in a planar set of n points that are ε-similar to a triangle T.

Overview

Definition

Let T, T' be triangles with angles $\alpha \leq \beta \leq \gamma$ and $\alpha' \leq \beta' \leq \gamma'$ respectively. The triangle T' is ε-*similar* to T if $|\alpha - \alpha'| < \varepsilon, |\beta - \beta'| < \varepsilon,$ and $|\gamma - \gamma'| < \varepsilon$.

Definition

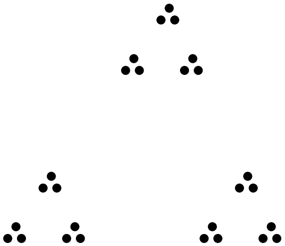
$h(n, T, \varepsilon) =$ The maximum number of triangles in a planar set of n points that are ε-similar to a triangle T.

Question (Bárány and Füredi (2019))

Given a triangle T, $n \in \mathbb{N}$ and $\varepsilon > 0$ sufficiently small, determine $h(n, T, \varepsilon)$.

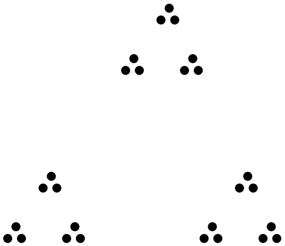
Lower bound construction

Let T be a triangle.



Lower bound construction

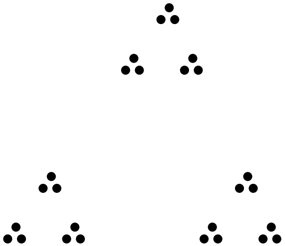
Let T be a triangle.



$$g(n) := \max\{abc + g(a) + g(b) + g(c) : a + b + c = n,\ \ a, b, c \in \mathbb{N}\}.$$
$$g(0) = g(1) = g(2) = 0.$$

Lower bound construction

Let T be a triangle.



$$g(n) := \max\{abc + g(a) + g(b) + g(c) : a + b + c = n,\ \ a, b, c \in \mathbb{N}\}.$$

$$g(0) = g(1) = g(2) = 0.$$

$$g(n) = \frac{1}{4}\binom{n}{3}(1 + o(1))$$

and if n is a power of 3, then $\quad g(n) = \frac{1}{24}(n^3 - n)$

Observation (Bárány and Füredi (2019))

For every triangle T and for every $\varepsilon > 0$, we have

$$h(n, T, \varepsilon) \geq g(n) = \frac{1}{4}\binom{n}{3}(1 + o(1)).$$

History

Observation (Bárány and Füredi (2019))

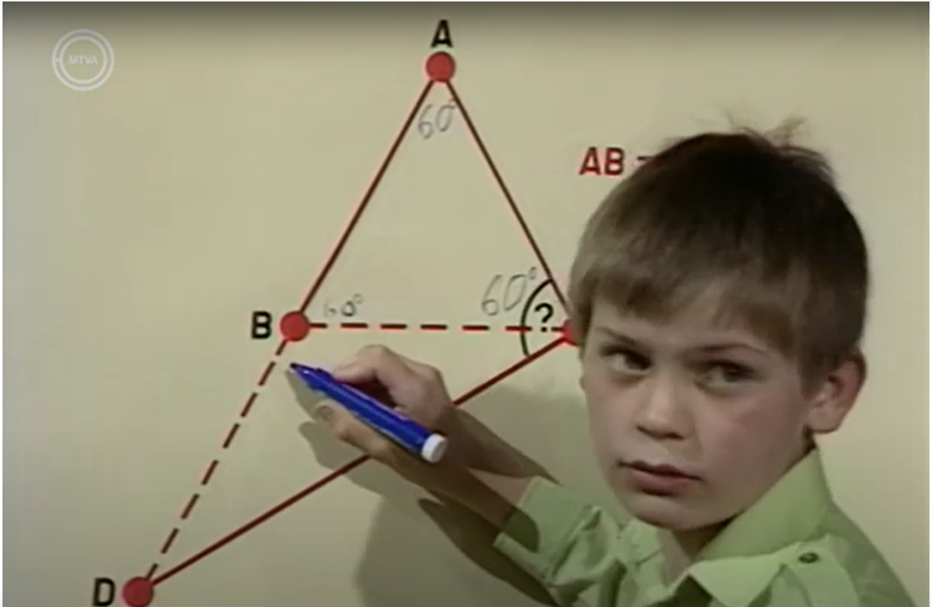
For every triangle T and for every $\varepsilon > 0$, we have

$$h(n, T, \varepsilon) \geq g(n) = \frac{1}{4}\binom{n}{3}(1 + o(1)).$$

Theorem (Bárány and Füredi (2019))

Let T be an equilateral triangle. There exists $\varepsilon_0 \geq 1^\circ$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all n we have $h(n, T, \varepsilon) = g(n)$.

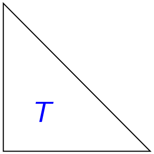
József Balogh 1986



Felix Christian Clemen

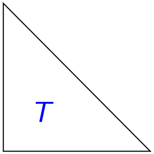
Lower bound construction for isosceles right triangle

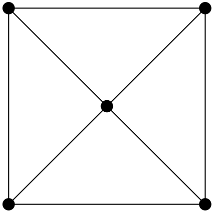
There are triangles T with $h(n, T, \varepsilon) > g(n)$.



Lower bound construction for isosceles right triangle

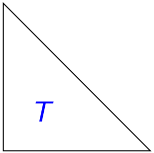
There are triangles T with $h(n, T, \varepsilon) > g(n)$.

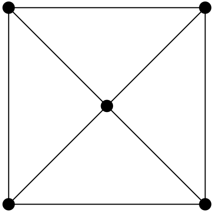




Lower bound construction for isosceles right triangle

There are triangles T with $h(n, T, \varepsilon) > g(n)$.





Observation (Bárány and Füredi (2019))

For T being the isosceles right triangle and for every $\varepsilon > 0$, we have

$$h(n, T, \varepsilon) \geq \frac{n^3}{6\sqrt{2}+6}(1+o(1)) \approx 0.414\binom{n}{3}(1+o(1)).$$

They found more triangle shapes T with $h(n, T, \varepsilon) > g(n)$.

Main result

Theorem (Bárány and Füredi (2019))

For almost every triangle T there is an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$,

$$h(n, T, \varepsilon) \leq 0.25072 \binom{n}{3} (1 + o(1)).$$

Theorem (Balogh, Clemen, Lidický (2021))

For almost every triangle T there is an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$,

$$h(n, T, \varepsilon) = \frac{1}{4} \binom{n}{3} (1 + o(1)).$$

Almost every triangle

- A triangle shape can be identified by points in $\mathbb{C}$
- Given a triangle shape T, there are up to 12 points in $\mathbb{C}$ which form T together with the points 0 and 1.

Almost every triangle

- A triangle shape can be identified by points in $\mathbb{C}$
- Given a triangle shape T, there are up to 12 points in $\mathbb{C}$ which form T together with the points 0 and 1.
- We say a statement about triangle shapes holds for *almost every triangle* if it holds for all triangle shapes except a set of Lebesgue measure 0.

Connection to Hypergraph Turán Problems

Let $P \subseteq \mathbb{R}^2$ be finite point set, T be a triangle.

- For $\varepsilon > 0$, let $G(P, T, \varepsilon)$ be the 3-graph with vertex set $V(G(P, T, \varepsilon)) = P$ and triples abc being an edge in $G(P, T, \varepsilon)$ iff abc forms a triangle ε-similar to T.
- A 3-graph H is called *forbidden* if $|V(H)| \leq 12$ and for almost every triangle T there exists an $\varepsilon = \varepsilon(T) > 0$ such that for every point set $P \subseteq \mathbb{R}^2$, $G(P, T, \varepsilon)$ is H-free.
- Denote $\mathcal{F}$ the family of all forbidden 3-graphs

Forbidden Subhypergraphs

Lemma (Bárány and Füredi, 2019)

The following hypergraphs are members of $\mathcal{F}$.

- $K_4^- = \{123, 124, 134\}$
- $C_5^- = \{123, 124, 135, 245\}$
- $C_5^+ = \{123, 234, 345, 356, 136\}$
- $L_2 = \{123, 124, 125, 136, 456\}$
- $L_3 = \{123, 124, 135, 256, 346\}$
- $L_4 = \{123, 124, 156, 256, 345\}$
- $L_5 = \{123, 124, 135, 146, 356\}$
- $L_6 = \{123, 124, 145, 346, 356\}$
- $P_7^- = \{123, 145, 167, 246, 257, 347\}$.

For almost all T, there exists $\varepsilon(T)$ such that

$$\begin{aligned} h(n, T, \varepsilon) &= \max_{P \subset \mathbb{R}^2, |P|=n} e(G(P, T, \varepsilon)) \\ &\leq \mathrm{ex}(n, \{K_4^-, C_5^-, C_5^+, L_2, L_3, L_4, L_5, L_6, P_7^-\}) \\ &\leq 0.25072 \binom{n}{3}(1 + o(1)). \end{aligned}$$

More forbidden subhypergraphs

Lemma

The following hypergraphs are members of $\mathcal{F}$.

- $L_7 = \{123, 124, 125, 136, 137, 458, 678\}$
- $L_8 = \{123, 124, 125, 136, 137, 468, 579, 289\}$
- $L_9 = \{123, 124, 125, 136, 237, 578, 469, 189\}$
- $L_{10} = \{123, 124, 125, 126, 137, 138, 239, 58a, 47b, 69c, abc\}$.

Definition

We call a 3-graph H on r vertices *dense* if there exists a vertex ordering $v_1, v_2, \ldots, v_r$ such that for every $3 \leq i \leq r-1$ there exists exactly one edge $e_i \in E(H[\{v_1, \ldots, v_i\}])$ containing v_i, and there exists exactly two edges e_r, e_r' containing v_r.

- A triangle shape T is represented by at most 12 points in $\mathbb{C}$:
  $\{z, 1-z, \frac{1}{z}, 1-\frac{1}{z}, \frac{1}{1-z}, \frac{z}{z-1}$ and their conjugates$\}$.

Finding forbidden subhypergraphs - a sketch of a sketch

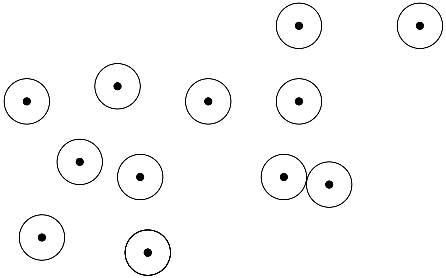
- A triangle shape T is represented by at most 12 points in $\mathbb{C}$: $\{z, 1-z, \frac{1}{z}, 1-\frac{1}{z}, \frac{1}{1-z}, \frac{z}{z-1} \text{and their conjugates}\}$.
- We will try to embed some $H \in \{L_7, L_8, L_9, L_{10}\}$ on points $p_1, p_2, p_3, \ldots, p_r \in \mathbb{C}$.

Finding forbidden subhypergraphs - a sketch of a sketch

- A triangle shape T is represented by at most 12 points in $\mathbb{C}$: $\{z, 1-z, \frac{1}{z}, 1-\frac{1}{z}, \frac{1}{1-z}, \frac{z}{z-1} \text{and their conjugates}\}$.
- We will try to embed some $H \in \{L_7, L_8, L_9, L_{10}\}$ on points $p_1, p_2, p_3, \ldots, p_r \in \mathbb{C}$.
- Assume $p_1 = 0, p_2 = 1$
- There are at most 12 ε'-balls to place p_3.
- There are at most 12^2 ε'-balls to place p_4.

Fix one of the 12^{r-3} possibilities to place the centers for the positions of $p_1, \ldots, p_{r-1}$.

Positions for p_r given e_r is an edge

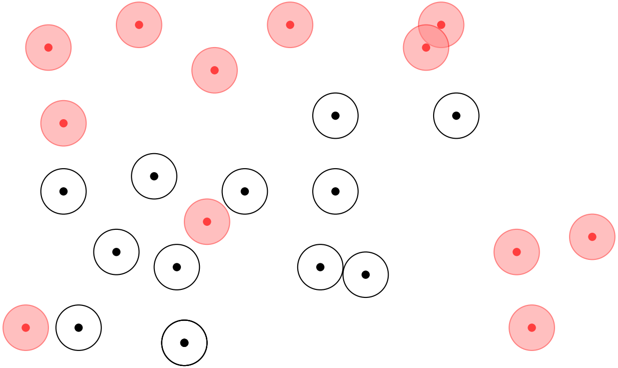


Finding forbidden subhypergraphs - a sketch of a sketch

Fix one of the 12^{r-3} possibilities to place the centers for the positions of $p_1, \ldots, p_{r-1}$.

Positions for p_r given e_r is an edge

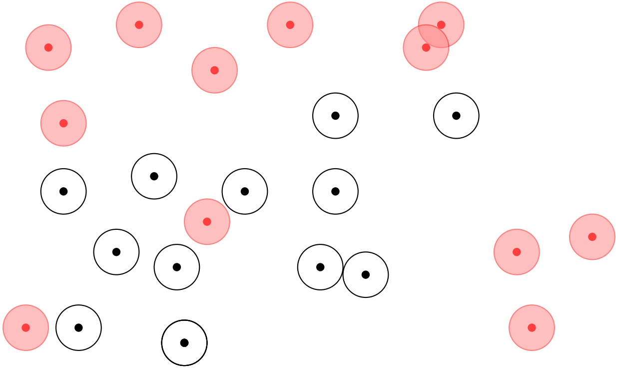
Positions for p_r given e'_r is an edge



Fix one of the 12^{r-3} possibilities to place the centers for the positions of $p_1, \ldots, p_{r-1}$.

Positions for p_r given e_r is an edge

Positions for p_r given e'_r is an edge



Intersection of red and black circles decides possibility of embedding. $\square$

Flag algebras

Seminal paper:
Razborov, Flag Algebras, *Journal of Symbolic Logic* **72** (2007), 1239–1282.
David P. Robbins Prize by AMS for Razborov in 2013

- Designed to attack extremal problems.
- The results are for the limit as graphs get very large.
- Calculating with densities of small induced graphs.

Flag algebras examples and our formulation

- Maximize K_2 subject to K_3-free
- Maximize C_5 subject to K_3-free
- Mimimize K_3 subject to $K_2 \geq p$
- Maximize $|E|$ subject to K_4^3-free
- Maximize $|E|$ subject to $\mathcal{F}$-free (our case)
- Maximize $|E|$ subject to $\mathcal{F}_7'$-free (really our case)

Weak stability

Lemma

Let $n \in \mathbb{N}$ *be sufficiently large and let* G *be an* $\mathcal{F}$*-free* 3*-graph on* n *vertices and* $|E(G)| \geq 1/24n^3(1+o(1))$ *edges. Then there exists an edge* $x_1x_2x_3 \in E(G)$ *such that for* n *large enough*

(i) *the neighborhoods* $N(x_1,x_2), N(x_2,x_3)$*, and* $N(x_1,x_3)$ *are pairwise disjoint,*

(ii) $\min\{|N(x_1,x_2)|, |N(x_2,x_3)|, |N(x_1,x_3)|\} \geq 0.26n$,

(iii) $|N(x_1,x_2)| + |N(x_2,x_3)| + |N(x_1,x_3)| \geq 0.988n$.

Proof sketch of weak stability via Flag Algebras

We want to find an edge $x_1x_2x_3$ such that

$$f(|A_1|, |A_2|, |A_3|) :=$$
$$|A_1||A_2| + |A_1||A_2| + |A_2||A_3| - \frac{1}{4}(|A_1|^2 + |A_2|^2 + |A_3|^2)$$

is large, where

$$A_1 := N(x_2, x_3), \quad A_2 := N(x_1, x_3), \quad A_3 := N(x_1, x_2),$$

Proof sketch of weak stability via Flag Algebras

We want to find an edge $x_1x_2x_3$ such that

$$f(|A_1|,|A_2|,|A_3|) :=$$
$$|A_1||A_2| + |A_1||A_2| + |A_2||A_3| - \frac{1}{4}(|A_1|^2 + |A_2|^2 + |A_3|^2)$$

is large, where

$$A_1 := N(x_2,x_3), \quad A_2 := N(x_1,x_3), \quad A_3 := N(x_1,x_2),$$

Pick an edge $x_1x_2x_3$ uniformly at random.

$$\mathbb{E}[f(|A_1|,|A_2|,|A_3|)] \geq \ldots \geq ((48t_{221} - 9t_{331})/40t_{111} + o(1))n^2 \geq 0.2213n^2$$

Proof sketch of weak stability via Flag Algebras

We want to find an edge $x_1x_2x_3$ such that

$$f(|A_1|,|A_2|,|A_3|) :=$$
$$|A_1||A_2| + |A_1||A_2| + |A_2||A_3| - \frac{1}{4}(|A_1|^2 + |A_2|^2 + |A_3|^2)$$

is large, where

$$A_1 := N(x_2,x_3), \quad A_2 := N(x_1,x_3), \quad A_3 := N(x_1,x_2),$$

Pick an edge $x_1x_2x_3$ uniformly at random.

$$0.25n^2 \geq \mathbb{E}[f(|A_1|,|A_2|,|A_3|)] \geq \ldots \geq ((48t_{221} - 9t_{331})/40t_{111} + o(1))n^2 \geq 0.2213n^2$$

Proof sketch of weak stability via Flag Algebras

We want to find an edge $x_1x_2x_3$ such that

$$f(|A_1|,|A_2|,|A_3|) :=$$
$$|A_1||A_2| + |A_1||A_2| + |A_2||A_3| - \frac{1}{4}(|A_1|^2 + |A_2|^2 + |A_3|^2)$$

is large, where

$$A_1 := N(x_2,x_3), \quad A_2 := N(x_1,x_3), \quad A_3 := N(x_1,x_2),$$

Pick an edge $x_1x_2x_3$ uniformly at random.

$$0.25n^2 \geq \mathbb{E}[f(|A_1|,|A_2|,|A_3|)] \geq \ldots \geq ((48t_{221} - 9t_{331})/40t_{111} + o(1))n^2 \geq 0.2213n^2$$

Thus, there exists an edge $x_1x_2x_3$ such that

$$\min\{|A_1|,|A_2|,|A_3|\} \geq 0.17n \quad \text{and} \quad |A_1|+|A_2|+|A_3| \geq 0.94n$$

$\square$

Proof sketch of weak stability via Flag Algebras

We want to find an edge $x_1x_2x_3$ such that

$$f(|A_1|,|A_2|,|A_3|) :=$$
$$|A_1||A_2| + |A_1||A_2| + |A_2||A_3| - \frac{1}{4}(|A_1|^2 + |A_2|^2 + |A_3|^2)$$

is large, where

$$A_1 := N(x_2,x_3), \quad A_2 := N(x_1,x_3), \quad A_3 := N(x_1,x_2),$$

Pick an edge $x_1x_2x_3$ uniformly at random.

$$0.25n^2 \geq \mathbb{E}[f(|A_1|,|A_2|,|A_3|)] \geq \ldots \geq ((48t_{221} - 9t_{331})/40t_{111} + o(1))n^2 \geq 0.2213n^2$$

Thus, there exists an edge $x_1x_2x_3$ such that

$$\min\{|A_1|,|A_2|,|A_3|\} \geq 0.17n \quad \text{and} \quad |A_1|+|A_2|+|A_3| \geq 0.94n$$

$\square$

$$\mathbb{E}[f] \geq 0.244n^2 \qquad \min\{|A_i|\} \geq 0.26n \qquad \sum |A_i| \geq 0.988n$$

The idea of our proof

- Assume G is $\mathcal{F}$-free 3-graph on n vertices with $|E(G)| \geq \frac{1}{4}\binom{n}{3}(1 + o(1))$ edges.

The idea of our proof

- Assume G is $\mathcal{F}$-free 3-graph on n vertices with $|E(G)| \geq \frac{1}{4}\binom{n}{3}(1+o(1))$ edges.
- Find an edge $x_1x_2x_3$ with large. $|N(x_1,x_2)| + |N(x_1,x_3)| + |N(x_2,x_3)|$

The idea of our proof

- Assume G is $\mathcal{F}$-free 3-graph on n vertices with $|E(G)| \geq \frac{1}{4}\binom{n}{3}(1 + o(1))$ edges.
- Find an edge $x_1x_2x_3$ with large. $|N(x_1, x_2)| + |N(x_1, x_3)| + |N(x_2, x_3)|$
- Use a cleaning technique to clean the "top layer".

CLAIM

We can partition $V(G) = X_1 \cup X_2 \cup X_3$ *with* $|X_i| \geq 0.26n$ *for* $i \in [3]$ *such that no triple* abc *with* $a, b \in X_i$ *and* $c \in X_j$ *for some* $i, j \in [3]$ *with* $i \neq j$ *forms an edge.*

Claim

We can partition $V(G) = X_1 \cup X_2 \cup X_3$ *with* $|X_i| \geq 0.26n$ *for* $i \in [3]$ *such that no triple* abc *with* $a, b \in X_i$ *and* $c \in X_j$ *for some* $i, j \in [3]$ *with* $i \neq j$ *forms an edge.*

Proof sketch: Main result.

$$e(G) \leq |X_1||X_2||X_3| + \sum_{i=1}^{3} e(G[X_i])$$

$\square$

Claim

We can partition $V(G) = X_1 \cup X_2 \cup X_3$ *with* $|X_i| \geq 0.26n$ *for* $i \in [3]$ *such that no triple* abc *with* $a, b \in X_i$ *and* $c \in X_j$ *for some* $i, j \in [3]$ *with* $i \neq j$ *forms an edge.*

Proof sketch: Main result.

We will prove by induction $\mathrm{ex}(n, \mathcal{F}) \leq \frac{1}{24}n^3 + Cn$.

$$e(G) \leq |X_1||X_2||X_3| + \sum_{i=1}^{3} e(G[X_i]) \leq |X_1||X_2||X_3| + Cn + \frac{1}{24}\sum_{i=1}^{3}|X_i|^3$$

$\square$

Optimization

Claim

We can partition $V(G) = X_1 \cup X_2 \cup X_3$ with $|X_i| \geq 0.26n$ for $i \in [3]$ such that no triple abc with $a, b \in X_i$ and $c \in X_j$ for some $i, j \in [3]$ with $i \neq j$ forms an edge.

Proof sketch: Main result.

We will prove by induction $\mathrm{ex}(n, \mathcal{F}) \leq \frac{1}{24}n^3 + Cn$.

$$e(G) \leq |X_1||X_2||X_3| + \sum_{i=1}^{3} e(G[X_i]) \leq |X_1||X_2||X_3| + Cn + \frac{1}{24}\sum_{i=1}^{3}|X_i|^3$$
$$\leq \frac{1}{24}n^3 + Cn,$$

$\square$

Optimization

Claim

We can partition $V(G) = X_1 \cup X_2 \cup X_3$ *with* $|X_i| \geq 0.26n$ *for* $i \in [3]$ *such that no triple* abc *with* $a, b \in X_i$ *and* $c \in X_j$ *for some* $i, j \in [3]$ *with* $i \neq j$ *forms an edge.*

Proof sketch: Main result.

We will prove by induction $\mathrm{ex}(n, \mathcal{F}) \leq \frac{1}{24}n^3 + Cn$.

$$
\begin{aligned}
e(G) &\leq |X_1||X_2||X_3| + \sum_{i=1}^{3} e(G[X_i]) \leq |X_1||X_2||X_3| + Cn + \frac{1}{24}\sum_{i=1}^{3}|X_i|^3 \\
&\leq \frac{1}{24}n^3 + Cn,
\end{aligned}
$$

where the maximum is obtained at $|X_i| = \frac{n}{3}(1 + o(1))$. $\square$

Completing the proof

Theorem (Balogh, C., Lidický (2021))

For almost every triangle T *there is an* $\varepsilon > 0$ *such that*

$$h(n, T, \varepsilon) = \frac{1}{4}\binom{n}{3}(1 + o(1)).$$

Proof.

$$h(n, T, \varepsilon) = \max_{P \subseteq \mathbb{R}^2, |P| = n} e(G(P, T, \varepsilon)) \le \mathrm{ex}(n, \mathcal{F}) \le \frac{1}{4}\binom{n}{3}(1 + o(1)).$$

$\square$

THEOREM (BALOGH, CLEMEN, LIDICKÝ (2021))

There exists n_0 *such that for all* $n \geq n_0$ *and for almost every triangle* T *there is an* $\varepsilon > 0$ *such that*

$$h(n, T, \varepsilon) = a \cdot b \cdot c + h(a, T, \varepsilon) + h(b, T, \varepsilon) + h(c, T, \varepsilon),$$

where $n = a + b + c$ *and* a, b, c *are as equal as possible.*

Almost all triangles - our results

Theorem (Balogh, Clemen, Lidický (2021))

There exists n_0 *such that for all* $n \geq n_0$ *and for almost every triangle* T *there is an* $\varepsilon > 0$ *such that*

$$h(n, T, \varepsilon) = a \cdot b \cdot c + h(a, T, \varepsilon) + h(b, T, \varepsilon) + h(c, T, \varepsilon),$$

where $n = a + b + c$ *and* a, b, c *are as equal as possible.*

Corollary (Balogh, Clemen, Lidický (2021))

Let n *be a power of* 3*. Then, for almost every triangle* T *there is an* $\varepsilon > 0$ *such that*

$$h(n, T, \varepsilon) = g(n) = \frac{1}{24}(n^3 - n).$$

Open Problems

Open Problems

- Consider point sets in $\mathbb{R}^3$ or even $\mathbb{R}^d$.

Open Problems

- Consider point sets in $\mathbb{R}^3$ or even $\mathbb{R}^d$.
- Determine $h(n, T, \varepsilon)$ for all T (and all ε small enough).

Open Problems

- Consider point sets in $\mathbb{R}^3$ or even $\mathbb{R}^d$.
- Determine $h(n, T, \varepsilon)$ for all T (and all ε small enough).
- Determine $\text{ex}(n, \mathcal{F})$ for a smaller family than $\mathcal{F}$

Open problem: $\mathbb{R}^3$ instead of $\mathbb{R}^2$

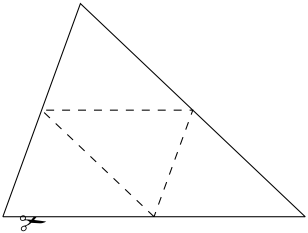


Figure: A cutout of a tetrahedron using an acute triangle on the left.

Open problem: $\mathbb{R}^3$ instead of $\mathbb{R}^2$

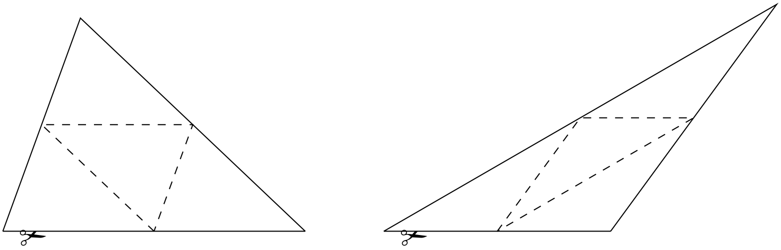


Figure: A cutout of a tetrahedron using an acute triangle on the left. A cutout not giving a tetrahedron coming from an obtuse triangle on the right.

Open problem: $\mathbb{R}^3$ instead of $\mathbb{R}^2$

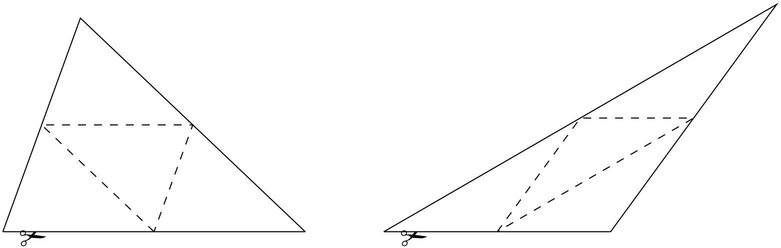


Figure: A cutout of a tetrahedron using an acute triangle on the left. A cutout not giving a tetrahedron coming from an obtuse triangle on the right.

Observation

Let $\varepsilon > 0$ and T be and acute triangle. Then, there exists a point set $P \subseteq \mathbb{R}^3$ with at least $\frac{2}{5}\binom{n}{3}(1 + o(1))$ triangles being ε-similar to T.

Open Problem: Smaller family than $\mathcal{F}$

Conjecture (Falgas-Ravry and Vaughan(2013))

$$\mathrm{ex}(n, \{K_4^-, C_5\}) = \frac{1}{4}\binom{n}{3}(1 + o(1)).$$

$L_2 = \{123, 124, 125, 136, 456\}$

Open Problem: Smaller family than $\mathcal{F}$

Conjecture (Falgas-Ravry and Vaughan(2013))

$$\mathrm{ex}(n, \{K_4^-, C_5\}) = \frac{1}{4}\binom{n}{3}(1 + o(1)).$$

Conjecture (Balogh, Clemen, Lidický (2021))

$$\mathrm{ex}(n, \{K_4^-, L_2\}) = \frac{1}{4}\binom{n}{3}(1 + o(1)).$$

$L_2 = \{123, 124, 125, 136, 456\}$

Thank you!

Thank you for your attention!

More detailed proof

- Denote $T_{i,j,k}$ the family of 3-graphs that are obtained from a complete 3-partite 3-graph with part sizes i, j and k by adding $\mathcal{F}$-free 3-graphs in each of the three parts.

More detailed proof

- Denote $T_{i,j,k}$ the family of 3-graphs that are obtained from a complete 3-partite 3-graph with part sizes i, j and k by adding $\mathcal{F}$-free 3-graphs in each of the three parts.
- The normalized number is $t_{i,j,k}$ is the number of subgraphs of G isomorphic to some $T \in T_{i,j,k}$ divided by $\binom{n}{i+j+k}$.

More detailed proof

- Denote $T_{i,j,k}$ the family of 3-graphs that are obtained from a complete 3-partite 3-graph with part sizes i, j and k by adding $\mathcal{F}$-free 3-graphs in each of the three parts.
- The normalized number is $t_{i,j,k}$ is the number of subgraphs of G isomorphic to some $T \in T_{i,j,k}$ divided by $\binom{n}{i+j+k}$.

$$\mathbb{E}[|A_1||A_2| + |A_1||A_2| + |A_2||A_3| - \frac{1}{4}(|A_1|^2 + |A_2|^2 + |A_3|^2)]$$

More detailed proof

- Denote $T_{i,j,k}$ the family of 3-graphs that are obtained from a complete 3-partite 3-graph with part sizes i, j and k by adding $\mathcal{F}$-free 3-graphs in each of the three parts.
- The normalized number is $t_{i,j,k}$ is the number of subgraphs of G isomorphic to some $T \in T_{i,j,k}$ divided by $\binom{n}{i+j+k}$.

$$\mathbb{E}[|A_1||A_2| + |A_1||A_2| + |A_2||A_3| - \frac{1}{4}(|A_1|^2 + |A_2|^2 + |A_3|^2)]$$
$$\geq \ldots \geq \frac{1}{7\,t_{2,2,1}}\left(3\,t_{3,3,1} + 3.5\,t_{3,2,2} - \,t_{4,2,1}\right)\binom{n-5}{2} + o(n^2)$$

More detailed proof

- Denote $T_{i,j,k}$ the family of 3-graphs that are obtained from a complete 3-partite 3-graph with part sizes i, j and k by adding $\mathcal{F}$-free 3-graphs in each of the three parts.
- The normalized number is $t_{i,j,k}$ is the number of subgraphs of G isomorphic to some $T \in T_{i,j,k}$ divided by $\binom{n}{i+j+k}$.

$$
\begin{aligned}
&\mathbb{E}[|A_1||A_2| + |A_1||A_2| + |A_2||A_3| - \frac{1}{4}(|A_1|^2 + |A_2|^2 + |A_3|^2)] \\
&\geq \ldots \geq \frac{1}{7\, t_{2,2,1}} (3\, t_{3,3,1} + 3.5\, t_{3,2,2} - \, t_{4,2,1}) \binom{n-5}{2} + o(n^2) \\
&\geq \frac{1.2814228}{14 \cdot 0.37502377} n^2 > 0.24406 n^2.
\end{aligned}
$$

Proof of weak stability via Flag Algebras

Let X be the subhypergraph on 5 vertices $x_1, x_1', x_2, x_2', x_3$ with edges $x_1x_2x_3, x_1x_2'x_3, x_1'x_2x_3, x_1'x_2'x_3$.

Proof of weak stability via Flag Algebras

Let X be the subhypergraph on 5 vertices $x_1, x_1', x_2, x_2', x_3$ with edges $x_1x_2x_3, x_1x_2'x_3, x_1'x_2x_3, x_1'x_2'x_3$. Denote

$$A_1 := N(x_2, x_3) \cap N(x_2', x_3), \quad A_2 := N(x_1, x_3) \cap N(x_1', x_3),$$
$$A_3 := N(x_1, x_2) \cap N(x_1', x_2) \cap N(x_1, x_2') \cap N(x_1', x_2')$$

Let X be the subhypergraph on 5 vertices $x_1, x_1', x_2, x_2', x_3$ with edges $x_1x_2x_3, x_1x_2'x_3, x_1'x_2x_3, x_1'x_2'x_3$. Denote

$$A_1 := N(x_2, x_3) \cap N(x_2', x_3), \quad A_2 := N(x_1, x_3) \cap N(x_1', x_3),$$
$$A_3 := N(x_1, x_2) \cap N(x_1', x_2) \cap N(x_1, x_2') \cap N(x_1', x_2')$$

and

$$f(|A_1|, |A_2|, |A_3|) :=$$
$$|A_1||A_2| + |A_1||A_2| + |A_2||A_3| - \frac{1}{4}(|A_1|^2 + |A_2|^2 + |A_3|^2)$$

Proof of weak stability via Flag Algebras

Let X be the subhypergraph on 5 vertices $x_1, x_1', x_2, x_2', x_3$ with edges $x_1x_2x_3, x_1x_2'x_3, x_1'x_2x_3, x_1'x_2'x_3$. Denote

$$A_1 := N(x_2, x_3) \cap N(x_2', x_3), \quad A_2 := N(x_1, x_3) \cap N(x_1', x_3),$$
$$A_3 := N(x_1, x_2) \cap N(x_1', x_2) \cap N(x_1, x_2') \cap N(x_1', x_2')$$

and

$$f(|A_1|, |A_2|, |A_3|) :=$$
$$|A_1||A_2| + |A_1||A_2| + |A_2||A_3| - \frac{1}{4}(|A_1|^2 + |A_2|^2 + |A_3|^2)$$

Pick X uniformly at random. We will lower bound

$$\mathbb{E}[f(|A_1|, |A_2|, |A_3|)] \geq \ldots \geq 0.24406n^2.$$

Proof of weak stability via Flag Algebras

Let X be the subhypergraph on 5 vertices $x_1, x_1', x_2, x_2', x_3$ with edges $x_1x_2x_3, x_1x_2'x_3, x_1'x_2x_3, x_1'x_2'x_3$. Denote

$$A_1 := N(x_2, x_3) \cap N(x_2', x_3), \quad A_2 := N(x_1, x_3) \cap N(x_1', x_3),$$
$$A_3 := N(x_1, x_2) \cap N(x_1', x_2) \cap N(x_1, x_2') \cap N(x_1', x_2')$$

and

$$f(|A_1|, |A_2|, |A_3|) :=$$
$$|A_1||A_2| + |A_1||A_2| + |A_2||A_3| - \frac{1}{4}(|A_1|^2 + |A_2|^2 + |A_3|^2)$$

Pick X uniformly at random. We will lower bound

$$0.25n^2 \geq \mathbb{E}[f(|A_1|, |A_2|, |A_3|)] \geq \ldots \geq 0.24406n^2.$$

Proof of weak stability via Flag Algebras

Let X be the subhypergraph on 5 vertices $x_1, x_1', x_2, x_2', x_3$ with edges $x_1x_2x_3, x_1x_2'x_3, x_1'x_2x_3, x_1'x_2'x_3$. Denote

$$A_1 := N(x_2, x_3) \cap N(x_2', x_3), \quad A_2 := N(x_1, x_3) \cap N(x_1', x_3),$$
$$A_3 := N(x_1, x_2) \cap N(x_1', x_2) \cap N(x_1, x_2') \cap N(x_1', x_2')$$

and

$$f(|A_1|, |A_2|, |A_3|) :=$$
$$|A_1||A_2| + |A_1||A_2| + |A_2||A_3| - \frac{1}{4}(|A_1|^2 + |A_2|^2 + |A_3|^2)$$

Pick X uniformly at random. We will lower bound

$$0.25n^2 \geq \mathbb{E}[f(|A_1|, |A_2|, |A_3|)] \geq \ldots \geq 0.24406n^2.$$

Thus, there exists a subhypergraph X such that

$$\min\{|A_1|, |A_2|, |A_3|\} \geq 0.26n \quad \text{and} \quad |A_1| + |A_2| + |A_3| \geq 0.988n$$

$\square$

More forbidden subhypergraphs

Lemma

The following hypergraphs are members of $\mathcal{F}$.

- $L_7 = \{123, 124, 125, 136, 137, 458, 678\}$
- $L_8 = \{123, 124, 125, 136, 137, 468, 579, 289\}$
- $L_9 = \{123, 124, 125, 136, 237, 578, 469, 189\}$
- $L_{10} = \{123, 124, 125, 126, 137, 138, 239, 58a, 47b, 69c, abc\}$.

Definition

We call a 3-graph H on r vertices *dense* if there exists a vertex ordering $v_1, v_2, \ldots, v_r$ such that for every $3 \leq i \leq r-1$ there exists exactly one edge $e_i \in E(H[\{v_1, \ldots, v_i\}])$ containing v_i, and there exists exactly two edges e_r, e'_r containing v_r.

How to find forbidden subhypergraphs - a sketch

- A triangle shape T is represented by at most 12 points in $\mathbb{C}$: $\{z, 1-z, \frac{1}{z}, 1-\frac{1}{z}, \frac{1}{1-z}, \frac{z}{z-1}$ and their conjugates$\}$.

How to find forbidden subhypergraphs - a sketch

- A triangle shape T is represented by at most 12 points in $\mathbb{C}$:
  $\{z, 1-z, \frac{1}{z}, 1-\frac{1}{z}, \frac{1}{1-z}, \frac{z}{z-1}$ and their conjugates$\}$.
- We will try to embed some $H \in \{L_7, L_8, L_9, L_{10}\}$ on points $p_1, p_2, p_3, \ldots, p_r \in \mathbb{C}$.

How to find forbidden subhypergraphs - a sketch

- A triangle shape T is represented by at most 12 points in $\mathbb{C}$: $\{z, 1-z, \frac{1}{z}, 1-\frac{1}{z}, \frac{1}{1-z}, \frac{z}{z-1}$ and their conjugates$\}$.
- We will try to embed some $H \in \{L_7, L_8, L_9, L_{10}\}$ on points $p_1, p_2, p_3, \ldots, p_r \in \mathbb{C}$.
- Assume $p_1 = 0, p_2 = 1$

How to find forbidden subhypergraphs - a sketch

- A triangle shape T is represented by at most 12 points in $\mathbb{C}$: $\{z, 1-z, \frac{1}{z}, 1-\frac{1}{z}, \frac{1}{1-z}, \frac{z}{z-1}\text{and their conjugates}\}$.
- We will try to embed some $H \in \{L_7, L_8, L_9, L_{10}\}$ on points $p_1, p_2, p_3, \ldots, p_r \in \mathbb{C}$.
- Assume $p_1 = 0, p_2 = 1$
- There are at most 12 ε'-balls to place p_3.

How to find forbidden subhypergraphs - a sketch

- A triangle shape T is represented by at most 12 points in $\mathbb{C}$: $\{z, 1-z, \frac{1}{z}, 1-\frac{1}{z}, \frac{1}{1-z}, \frac{z}{z-1} \text{and their conjugates}\}$.
- We will try to embed some $H \in \{L_7, L_8, L_9, L_{10}\}$ on points $p_1, p_2, p_3, \ldots, p_r \in \mathbb{C}$.
- Assume $p_1 = 0, p_2 = 1$
- There are at most 12 ε'-balls to place p_3.
- There are at most 12^2 ε'-balls to place p_4.

How to find forbidden subhypergraphs - a sketch

- A triangle shape T is represented by at most 12 points in $\mathbb{C}$: $\{z, 1-z, \frac{1}{z}, 1-\frac{1}{z}, \frac{1}{1-z}, \frac{z}{z-1} \text{and their conjugates}\}$.
- We will try to embed some $H \in \{L_7, L_8, L_9, L_{10}\}$ on points $p_1, p_2, p_3, \ldots, p_r \in \mathbb{C}$.
- Assume $p_1 = 0, p_2 = 1$
- There are at most 12 ε'-balls to place p_3.
- There are at most 12^2 ε'-balls to place p_4.
- There are at most 12^{r-3} ε'-balls to place p_{r-1}.

How to find forbidden subhypergraphs - a sketch

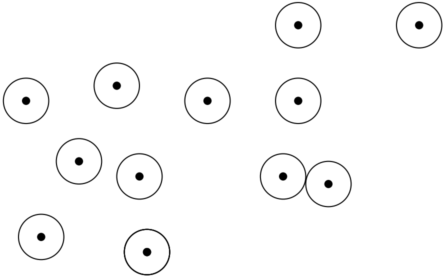
- A triangle shape T is represented by at most 12 points in $\mathbb{C}$: $\{z, 1-z, \frac{1}{z}, 1-\frac{1}{z}, \frac{1}{1-z}, \frac{z}{z-1}\text{and their conjugates}\}$.
- We will try to embed some $H \in \{L_7, L_8, L_9, L_{10}\}$ on points $p_1, p_2, p_3, \ldots, p_r \in \mathbb{C}$.
- Assume $p_1 = 0, p_2 = 1$
- There are at most 12 ε'-balls to place p_3.
- There are at most 12^2 ε'-balls to place p_4.
- There are at most 12^{r-3} ε'-balls to place p_{r-1}.
- Each of the centers of the balls are rational functions in z.

How to find forbidden subhypergraphs - a sketch

- A triangle shape T is represented by at most 12 points in $\mathbb{C}$: $\{z, 1-z, \frac{1}{z}, 1-\frac{1}{z}, \frac{1}{1-z}, \frac{z}{z-1} \text{and their conjugates}\}$.
- We will try to embed some $H \in \{L_7, L_8, L_9, L_{10}\}$ on points $p_1, p_2, p_3, \ldots, p_r \in \mathbb{C}$.
- Assume $p_1 = 0, p_2 = 1$
- There are at most 12 ε'-balls to place p_3.
- There are at most 12^2 ε'-balls to place p_4.
- There are at most 12^{r-3} ε'-balls to place p_{r-1}.
- Each of the centers of the balls are rational functions in z.
- Fix one of the 12^{r-3} possibilities to place the centers for the positions of $p_1, \ldots, p_{r-1}$.

How to find forbidden subhypergraphs - a sketch

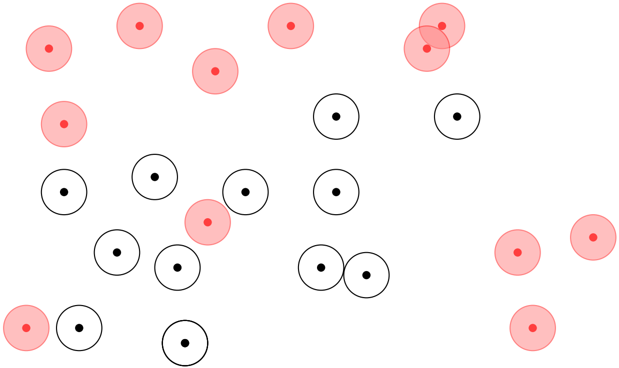
Positions for p_r given e_r is an edge



How to find forbidden subhypergraphs - a sketch

Positions for p_r given e_r is an edge

Positions for p_r given e'_r is an edge



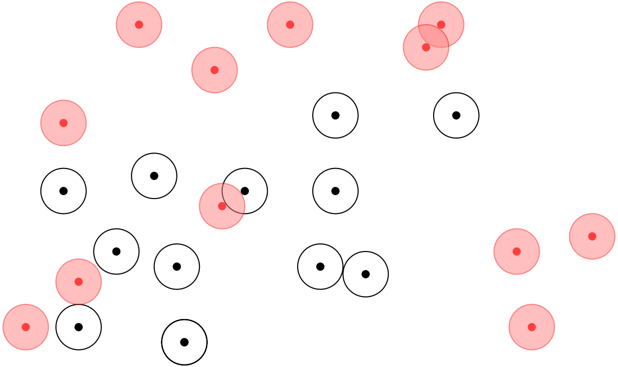
Situation 1: No intersection of red and black circles

$\rightarrow$ H cannot be embedded

How to find forbidden subhypergraphs - a sketch

Positions for p_r given e_r is an edge

Positions for p_r given e_r' is an edge



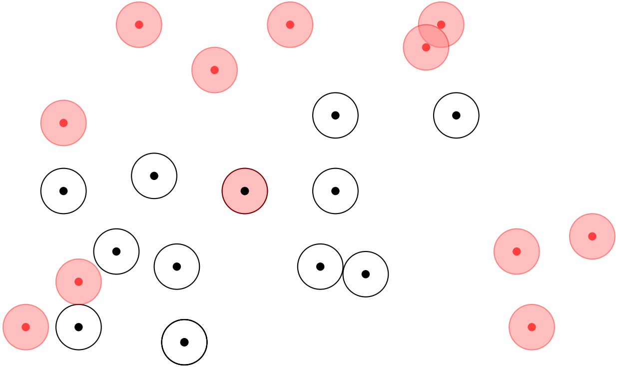
Situation 2: red and black circle intersect but do not have the same center

Reduce ε' and they do not intersect any longer $\rightarrow$ H cannot be embedded.

How to find forbidden subhypergraphs - a sketch

Positions for p_r given e_r is an edge

Positions for p_r given e_r' is an edge



Situation 3: A red and black circle has the same center

Claim: This only happens for a set of $z \in \mathbb{C}$ of Lebesgue measure 0.

How to find forbidden subhypergraphs - a sketch

- Two cycles have a matching center if a certain polynomial equation is satisfied.

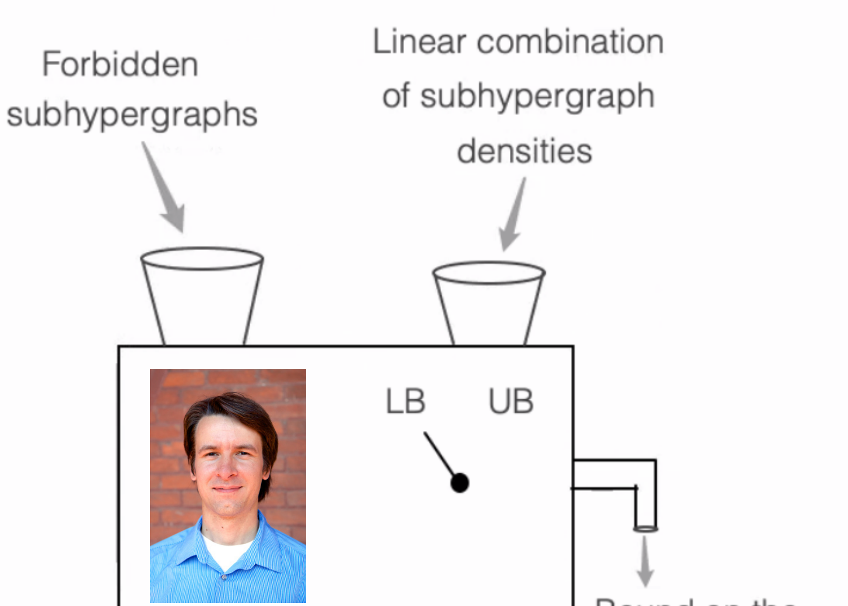
How to find forbidden subhypergraphs - a sketch

- Two cycles have a matching center if a certain polynomial equation is satisfied.
- Strategy: Show that there is a $z \in \mathbb{C}$ (We pick $z = \frac{1}{2} + i \cdot \frac{\sqrt{3}}{2}$) not satisfying any of these $12^{|V(H)|-1}$ equations. (This is lots of casework)

How to find forbidden subhypergraphs - a sketch

- Two cycles have a matching center if a certain polynomial equation is satisfied.
- Strategy: Show that there is a $z \in \mathbb{C}$ (We pick $z = \frac{1}{2} + i \cdot \frac{\sqrt{3}}{2}$) not satisfying any of these $12^{|V(H)|-1}$ equations. (This is lots of casework)
- The solution space of these equations must be of Lebesgue measure 0.

A picture of Flag Algebras



Applications (early incomplete list)

Author	Year	Application/Result
Razborov	2008	edge density vs. triangle density
Hladký, Král, Norin	2009	Bounds for the Caccetta-Haggvist conjecture
Razborov	2010	On 3-hypergraphs with forbidden 4-vertex configurations
Hatami, Hladký,Král,Norine,Razborov / Grzesik	2011	Erdős Pentagon problem
Hatami, Hladký, Král, Norin, Razborov	2012	Non-Three-Colourable Common Graphs Exist
Balogh, Hu, Lidický, Liu / Baber	2012	4-cycles in hypercubes
Das, Huang, Ma, Naves, Sudakov	2013	minimum number of k-cliques
Baber, Talbot	2013	A Solution to the 2/3 Conjecture
Falgas-Ravry, Vaughan	2013	Turán density of many 3-graphs
Cummings, Král, Pfender, Sperfeld, Treglown, Young	2013	Monochromatic triangles in 3-edge colored graphs
Kramer, Martin, Young	2013	Boolean lattice
Balogh, Hu, Lidický, Pikhurko, Udvari, Volec	2013	Monotone permutations
Norin, Zwols	2013	New bound on Zarankiewicz's conjecture
Huang, Linial, Naves, Peled, Sudakov	2014	3-local profiles of graphs
Balogh, Hu, Lidický, Pfender, Volec, Young	2014	Rainbow triangles in 3-edge colored graphs
Balogh, Hu, Lidický, Pfender	2014	Induced density of C_5
Goaoc, Hubard, de Verclos, Sereni, Volec	2014	Order type and density of convex subsets
Coregliano, Razborov	2015	Tournaments
Alon, Naves, Sudakov	2015	Phylogenetic trees
...	...	...