Maximum Number of Almost Similar Triangles in the Plane

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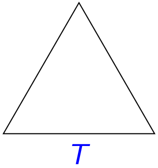
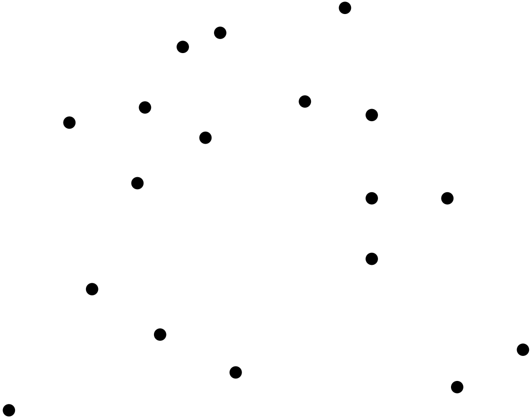
Joint Vanderbilt/MTSU Graph Theory and Combinatorics Seminar

April 27, 2022

Problem

Let T be a triangle and $n \in \mathbb{N}$ fixed.
Which n points in $\mathbb{R}^2$ maximize the number of triangles similar to T?





T_1 and T_2 are ε-*similar* if their inner angles differ by at most ε.
(OK to move, scale, rotate, ε-perturb)

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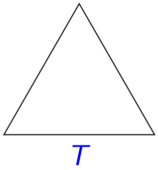

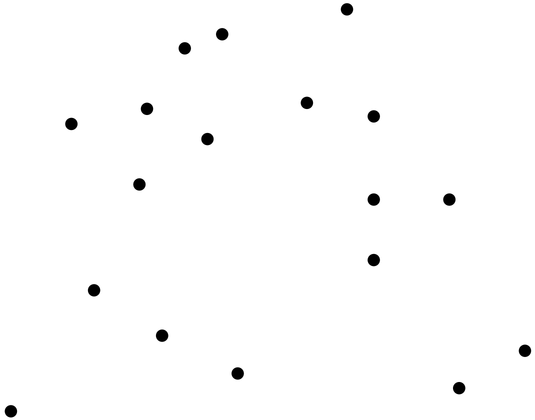

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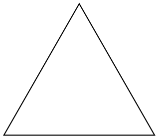

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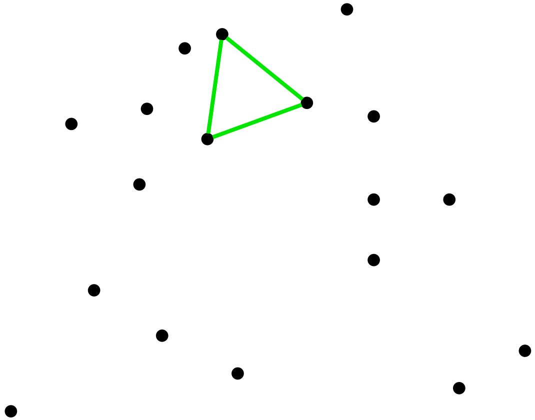

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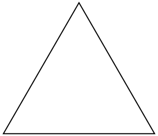

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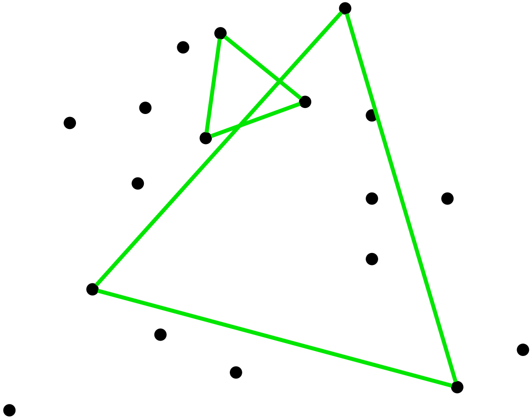

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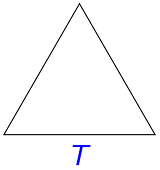

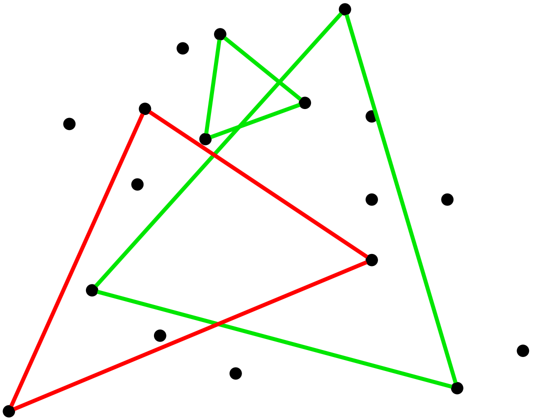

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Definition

Let T, T' be triangles with angles $\alpha \leq \beta \leq \gamma$ and $\alpha' \leq \beta' \leq \gamma'$ respectively. The triangle T' is *ε-similar* to T if $|\alpha - \alpha'| < \varepsilon, |\beta - \beta'| < \varepsilon,$ and $|\gamma - \gamma'| < \varepsilon$.

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$h(n, T, \varepsilon) =$ The maximum number of triangles in a planar set of n points that are ε-similar to a triangle T.

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Question (Bárány and Füredi (2019))

Given a triangle T, $n \in \mathbb{N}$ and $\varepsilon > 0$ sufficiently small, determine $h(n, T, \varepsilon)$.

Lower bound construction

Let T be a triangle and $n \in \mathbb{N}$ fixed.
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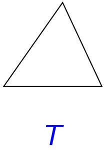

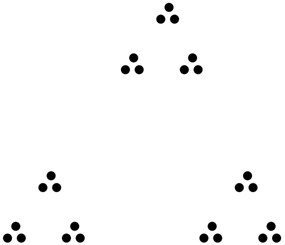

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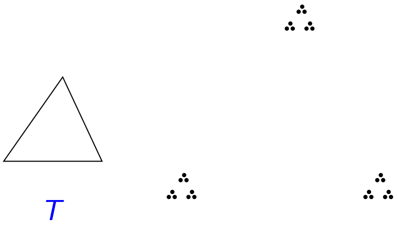

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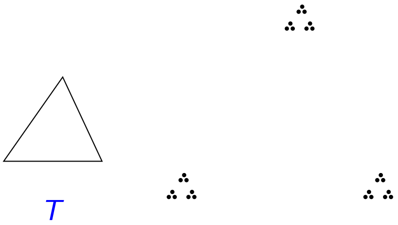

$$g(n) := \max\{abc + g(a) + g(b) + g(c) : a + b + c = n,\ a, b, c \in \mathbb{N}\}$$

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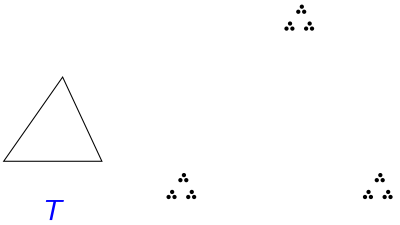

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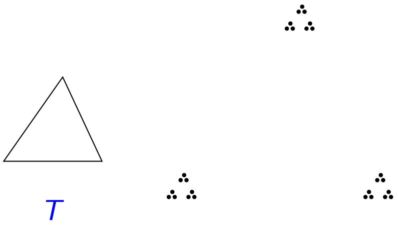

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$h(n, T, \varepsilon) :=$ max # of ε-*similar* triangles to T, it is at least $g(n) = \frac{1}{4}\binom{n}{3}(1 + o(1))$.

Observation (Bárány and Füredi (2019))

For every triangle T and for every $\varepsilon > 0$, we have

$$h(n, T, \varepsilon) \geq g(n) = \frac{1}{4}\binom{n}{3}(1 + o(1)).$$

History

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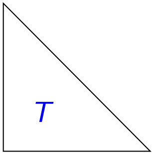
Theorem (Bárány and Füredi (2019))

Let T be an equilateral triangle. There exists $\varepsilon_0 \geq 1^\circ$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all n we have $h(n, T, \varepsilon) = g(n)$.

Lower bound construction for isosceles right triangle

There are triangles T with $h(n, T, \varepsilon) > g(n)$.



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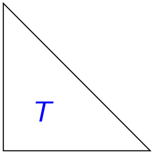

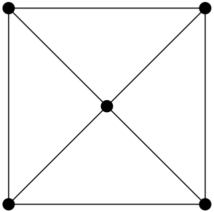

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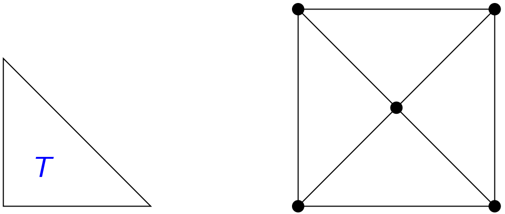

Observation (Bárány and Füredi (2019))

For T being the isosceles right triangle and for every $\varepsilon > 0$, we have

$$h(n, T, \varepsilon) \geq \frac{n^3}{6\sqrt{2}+6}(1+o(1)) \approx 0.414\binom{n}{3}(1+o(1)).$$

They found more triangle shapes T with $h(n, T, \varepsilon) > g(n)$.

Main result

Theorem (Bárány and Füredi (2019))

For almost every triangle T *there is an* $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon \leq \varepsilon_0$,

$$h(n, T, \varepsilon) \leq 0.25072 \binom{n}{3}(1 + o(1)).$$

Theorem (Balogh, Clemen, Lidický (2022))

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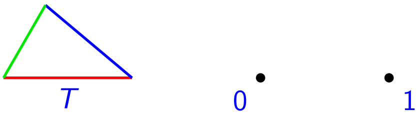
Almost every triangle T

- T can be identified by points in $\mathbb{C}$
- Given a triangle T, there are up to 12 points $c \in \mathbb{C}$ where $\{0, 1, c\}$ is 0-similar to T.



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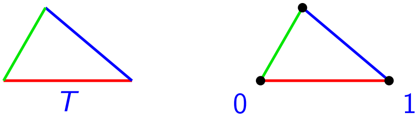

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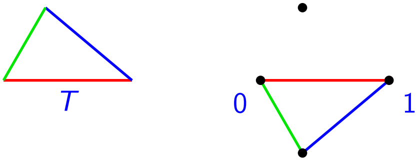

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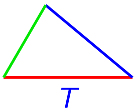

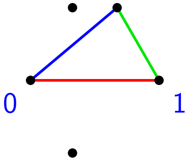

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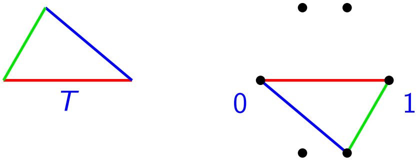

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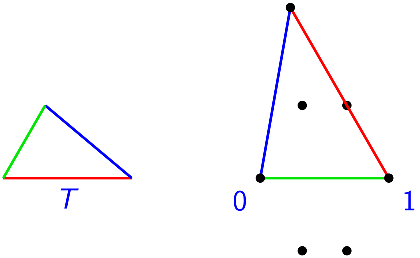

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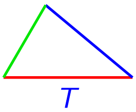

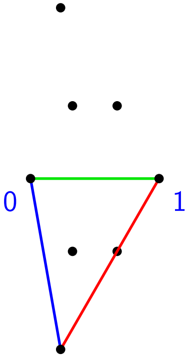

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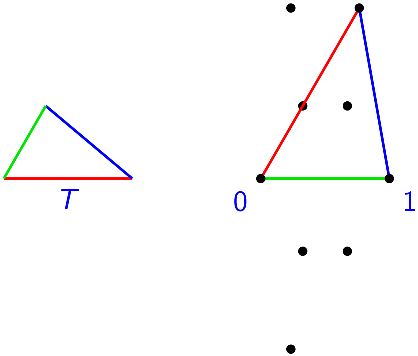

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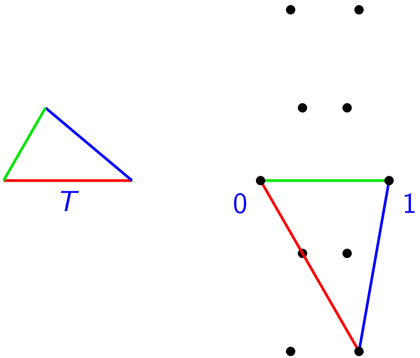

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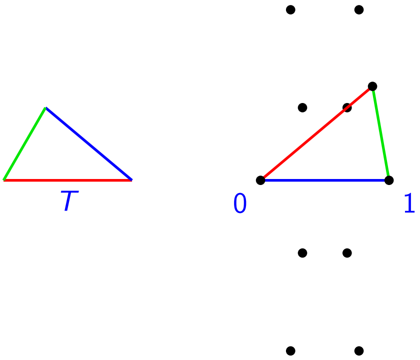

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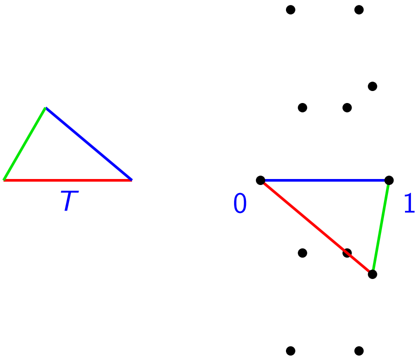

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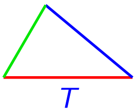

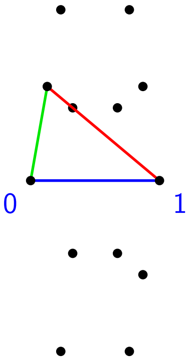

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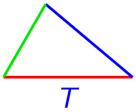

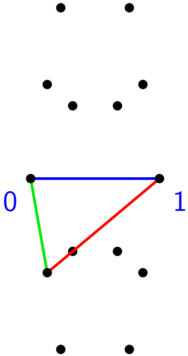

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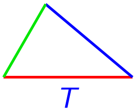
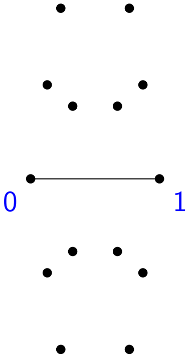
- Given a triangle T, there are up to 12 points $c \in \mathbb{C}$ where $\{0, 1, c\}$ is 0-similar to T.
- We say a statement about triangle shapes holds for *almost every triangle* if it holds for all triangle shapes except a set of Lebesgue measure 0.

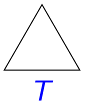


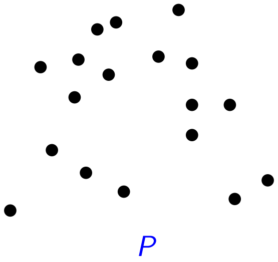


Connection to Hypergraph Turán Problems

Let $P \subseteq \mathbb{R}^2$ be finite point set, T be a triangle.

- For $\varepsilon > 0$, let $G(P, T, \varepsilon)$ be the 3-graph with vertex set $V(G(P, T, \varepsilon)) = P$ and triples abc being an edge in $G(P, T, \varepsilon)$ iff abc forms a triangle ε-similar to T.
- A 3-graph H is called *forbidden* if $|V(H)| \leq 12$ and for almost every triangle T there exists an $\varepsilon = \varepsilon(T) > 0$ such that for every point set $P \subseteq \mathbb{R}^2$, $G(P, T, \varepsilon)$ is H-free.
- Denote $\mathcal{F}$ the family of all forbidden 3-graphs
- $\mathcal{F}$ contains K_4^-, C_5^- and many others.
  Bárány and Füredi developed a testing procedure.





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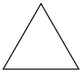

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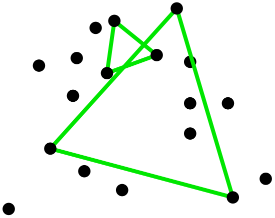

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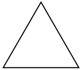
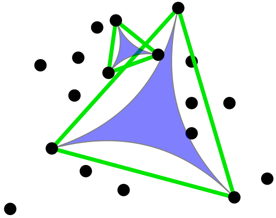
Let $P \subseteq \mathbb{R}^2$ be finite point set, T be a triangle.

- For $\varepsilon > 0$, let $G(P, T, \varepsilon)$ be the 3-graph with vertex set $V(G(P, T, \varepsilon)) = P$ and triples abc being an edge in $G(P, T, \varepsilon)$ iff abc forms a triangle ε-similar to T.
- A 3-graph H is called *forbidden* if $|V(H)| \leq 12$ and for almost every triangle T there exists an $\varepsilon = \varepsilon(T) > 0$ such that for every point set $P \subseteq \mathbb{R}^2$, $G(P, T, \varepsilon)$ is H-free.
- Denote $\mathcal{F}$ the family of all forbidden 3-graphs
- $\mathcal{F}$ contains K_4^-, C_5^- and many others.
  Bárány and Füredi developed a testing procedure.



T



P

Lemma (Bárány and Füredi, 2019)

The following hypergraphs are members of $\mathcal{F}$.

- $K_4^- = \{123, 124, 134\}$
- $C_5^- = \{123, 124, 135, 245\}$
- $C_5^+ = \{123, 234, 345, 356, 136\}$
- $L_2 = \{123, 124, 125, 136, 456\}$
- $L_3 = \{123, 124, 135, 256, 346\}$
- $L_4 = \{123, 124, 156, 256, 345\}$
- $L_5 = \{123, 124, 135, 146, 356\}$
- $L_6 = \{123, 124, 145, 346, 356\}$
- $P_7^- = \{123, 145, 167, 246, 257, 347\}$.

For almost all T, there exists $\varepsilon(T)$ such that

$$\begin{aligned} h(n, T, \varepsilon) &= \max_{P \subset \mathbb{R}^2, |P|=n} e(G(P, T, \varepsilon)) \\ &\leq \mathrm{ex}(n, \{K_4^-, C_5^-, C_5^+, L_2, L_3, L_4, L_5, L_6, P_7^-\}) \\ &\leq 0.25072 \binom{n}{3} (1 + o(1)). \end{aligned}$$

More forbidden subhypergraphs

Lemma

The following hypergraphs are members of $\mathcal{F}$.

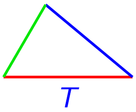
- $L_7 = \{123, 124, 125, 136, 137, 458, 678\}$
- $L_8 = \{123, 124, 125, 136, 137, 468, 579, 289\}$
- $L_9 = \{123, 124, 125, 136, 237, 578, 469, 189\}$
- $L_{10} = \{123, 124, 125, 126, 137, 138, 239, 58a, 47b, 69c, abc\}$.

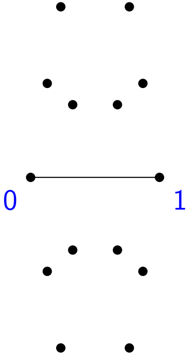
Definition

We call a 3-graph H on r vertices *dense* if there exists a vertex ordering $v_1, v_2, \ldots, v_r$ such that for every $3 \leq i \leq r-1$ there exists exactly one edge $e_i \in E(H[\{v_1, \ldots, v_i\}])$ containing v_i, and there exists exactly two edges e_r, e_r' containing v_r.

Finding forbidden subhypergraphs - a sketch of a sketch

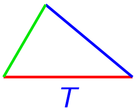
- A triangle shape T is represented by at most 12 points in $\mathbb{C}$.

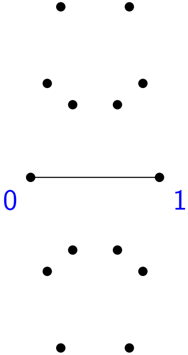




Finding forbidden subhypergraphs - a sketch of a sketch

- A triangle shape T is represented by at most 12 points in $\mathbb{C}$.
- We try to embed some $H \in \{L_7, L_8, L_9, L_{10}\}$ on points $p_1, p_2, p_3, \ldots, p_r \in \mathbb{C}$.
- $L_7 = \{123, 134, 125, 126, 137, 568, 478\}$





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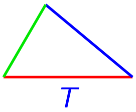
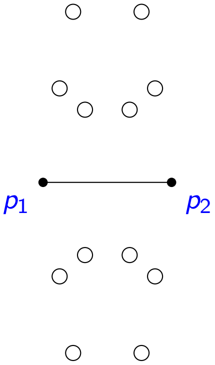
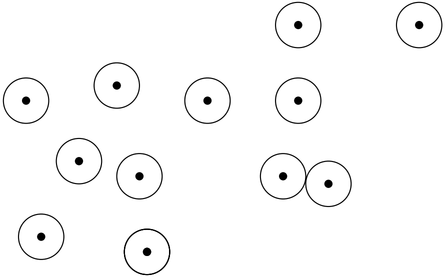
- $L_7 = \{123, 134, 125, 126, 137, 568, 478\}$
- Assume $p_1 = 0, p_2 = 1$
- There are at most 12 ε'-balls to place p_3.
- There are at most 12^2 ε'-balls to place p_4.





Fix one of the 12^{r-3} possibilities to place the centers for the positions of $p_1, \ldots, p_{r-1}$.

Positions for p_r given e_r is an edge



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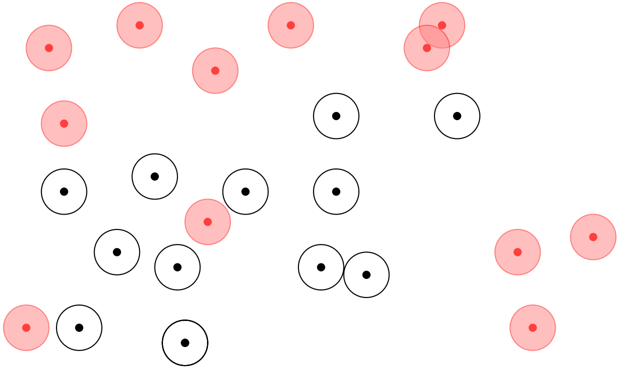

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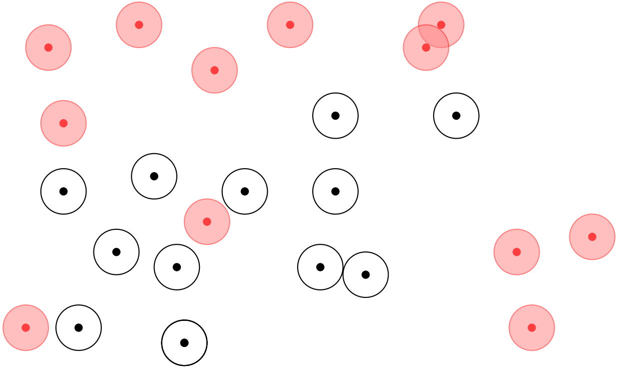


Intersection of red and black circles decides possibility of embedding. □

Flag algebras

Seminal paper:
Razborov, Flag Algebras, *Journal of Symbolic Logic* **72** (2007), 1239–1282.
David P. Robbins Prize by AMS for Razborov in 2013

- Designed to attack extremal problems.
- The results are for the limit as graphs get very large.
- Calculating with densities of small induced graphs.

Flag algebras examples and our formulation

- Maximize K_2 subject to K_3-free
- Maximize C_5 subject to K_3-free
- Mimimize K_3 subject to $K_2 \geq p$
- Maximize $|E|$ subject to K_4^3-free
- Maximize $|E|$ subject to $\mathcal{F}$-free (our case)
- Maximize $|E|$ subject to $\mathcal{F}_7'$-free (really our case)

Weak stability

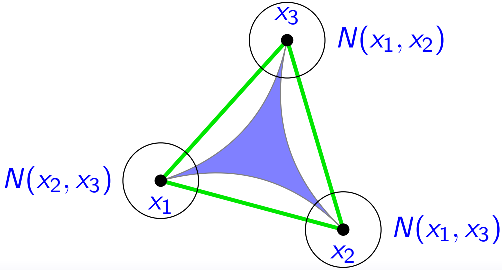
Lemma

Let $n \in \mathbb{N}$ *be sufficiently large and let* G *be an* $\mathcal{F}$*-free* 3*-graph on* n *vertices and* $|E(G)| \geq 1/24n^3(1+o(1))$ *edges. Then there exists an edge* $x_1x_2x_3 \in E(G)$ *such that for* n *large enough*

(i) *the neighborhoods* $N(x_1, x_2)$, $N(x_2, x_3)$, *and* $N(x_1, x_3)$ *are pairwise disjoint,*

(ii) $\min\{|N(x_1, x_2)|, |N(x_2, x_3)|, |N(x_1, x_3)|\} \geq 0.26n$,

(iii) $|N(x_1, x_2)| + |N(x_2, x_3)| + |N(x_1, x_3)| \geq 0.988n$.



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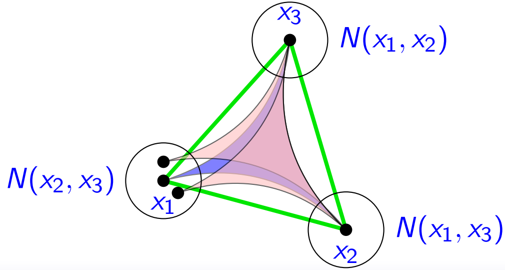

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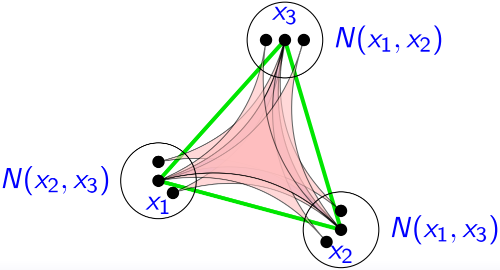

Proof sketch of weak stability via Flag Algebras

We want an edge $x_1x_2x_3$ such that $f(|A_1|, |A_2|, |A_3|) :=$

$$|A_1||A_2| + |A_1||A_2| + |A_2||A_3| - \frac{1}{4}(|A_1|^2 + |A_2|^2 + |A_3|^2)$$

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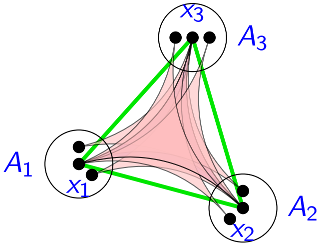

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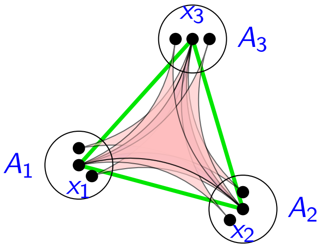

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Pick an edge $x_1x_2x_3$ uniformly at random.

$$\mathbb{E}[f(|A_1|, |A_2|, |A_3|)] \geq \ldots \geq ((48t_{221} - 9t_{331})/40t_{111} + o(1))n^2 \geq 0.2213n^2$$

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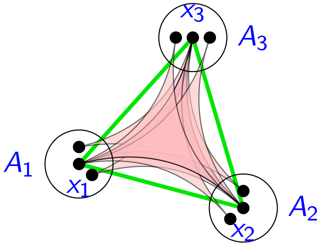

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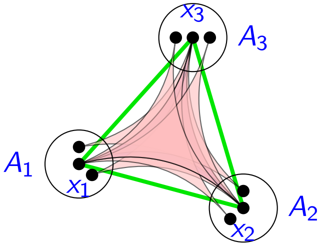

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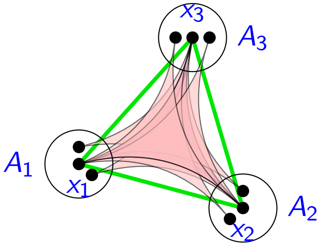

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$$\mathbb{E}[f] \geq 0.244n^2 \qquad \min\{|A_i|\} \geq 0.26n \qquad \sum |A_i| \geq 0.988n$$

The idea of our proof

- Assume G is $\mathcal{F}$-free 3-graph on n vertices with $|E(G)| \geq \frac{1}{4}\binom{n}{3}(1+o(1))$ edges.
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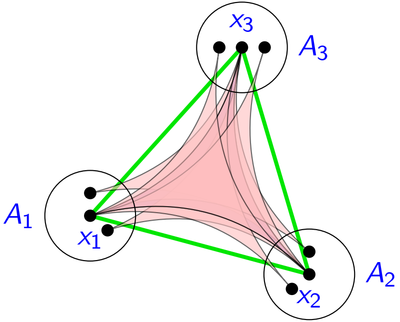

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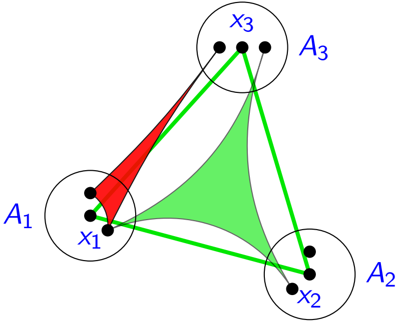
- Use a cleaning technique to clean the "top layer".



Claim

We can partition $V(G) = X_1 \cup X_2 \cup X_3$ *with* $|X_i| \geq 0.26n$ *for* $i \in [3]$ *such that no triple* abc *with* $a, b \in X_i$ *and* $c \in X_j$ *for some* $i, j \in [3]$ *with* $i \neq j$ *forms an edge.*

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Proof sketch: Main result.

$$e(G) \leq |X_1||X_2||X_3| + \sum_{i=1}^{3} e(G[X_i])$$

□

Optimization

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We will prove by induction $\mathrm{ex}(n, \mathcal{F}) \leq \frac{1}{24}n^3 + Cn$.

$$e(G) \leq |X_1||X_2||X_3| + \sum_{i=1}^{3} e(G[X_i]) \leq |X_1||X_2||X_3| + Cn + \frac{1}{24}\sum_{i=1}^{3} |X_i|^3$$

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\end{aligned}
$$

where the maximum is obtained at $|X_i| = \frac{n}{3}(1 + o(1))$. □

Completing the proof

Theorem (Balogh, Clemen, Lidický (2022))

For almost every triangle T there is an $\varepsilon > 0$ such that

$$h(n, T, \varepsilon) = \frac{1}{4}\binom{n}{3}(1 + o(1)).$$

Proof.

$$h(n, T, \varepsilon) = \max_{P \subseteq \mathbb{R}^2, |P| = n} e(G(P, T, \varepsilon)) \leq \mathrm{ex}(n, \mathcal{F}) \leq \frac{1}{4}\binom{n}{3}(1 + o(1)).$$

□

Almost all triangles - our results

Theorem (Balogh, Clemen, Lidický (2022))

There exists n_0 *such that for all* $n \geq n_0$ *and for almost every triangle* T *there is an* $\varepsilon > 0$ *such that*

$$h(n, T, \varepsilon) = a \cdot b \cdot c + h(a, T, \varepsilon) + h(b, T, \varepsilon) + h(c, T, \varepsilon),$$

where $n = a + b + c$ *and* a, b, c *are as equal as possible.*

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where $n = a + b + c$ and a, b, c are as equal as possible.

Corollary (Balogh, Clemen, Lidický (2022))

Let n be a power of 3. Then, for almost every triangle T there is an $\varepsilon > 0$ such that

$$h(n, T, \varepsilon) = g(n) = \frac{1}{24}(n^3 - n).$$

Open Problems

- Consider point sets in $\mathbb{R}^3$ or even $\mathbb{R}^d$.

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- Determine $h(n, T, \varepsilon)$ for all T (and all ε small enough).
- Determine $\mathrm{ex}(n, \mathcal{F})$ for a smaller family than $\mathcal{F}$

Open problem: $\mathbb{R}^3$ instead of $\mathbb{R}^2$

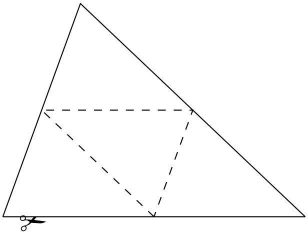

Figure: A cutout of a tetrahedron using an acute triangle on the left.

Open problem: $\mathbb{R}^3$ instead of $\mathbb{R}^2$

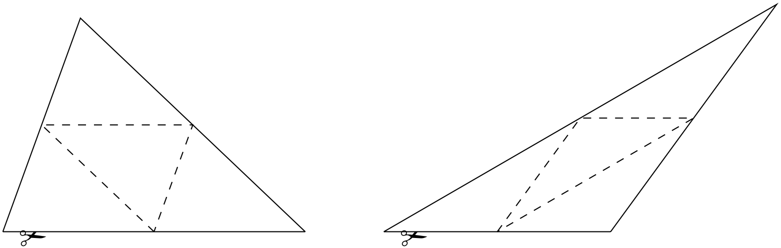

Figure: A cutout of a tetrahedron using an acute triangle on the left. A cutout not giving a tetrahedron coming from an obtuse triangle on the right.

Open problem: $\mathbb{R}^3$ instead of $\mathbb{R}^2$

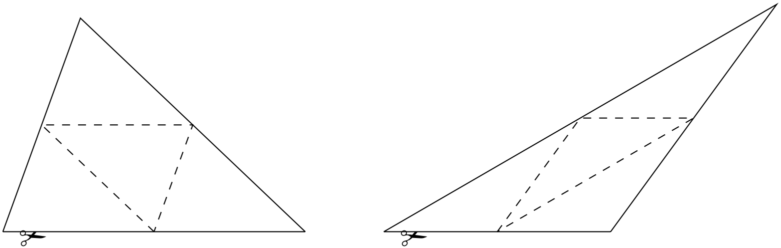

Figure: A cutout of a tetrahedron using an acute triangle on the left. A cutout not giving a tetrahedron coming from an obtuse triangle on the right.

Observation

Let $\varepsilon > 0$ and T be and acute triangle. Then, there exists a point set $P \subseteq \mathbb{R}^3$ with at least $\frac{2}{5}\binom{n}{3}(1 + o(1))$ triangles being ε-similar to T.

Open Problem: Smaller family than $\mathcal{F}$

Conjecture (Falgas-Ravry and Vaughan(2013))

$$\mathrm{ex}(n, \{K_4^-, C_5\}) = \frac{1}{4}\binom{n}{3}(1 + o(1)).$$

Conjecture (Vaughan (20??))

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Conjecture (Balogh, Clemen, Lidický (2022))

$$\mathrm{ex}(n, \{K_4^-, L_2\}) = \frac{1}{4}\binom{n}{3}(1 + o(1)).$$

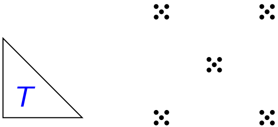
$L_2 = \{123, 124, 125, 136, 456\}$

Thank you!

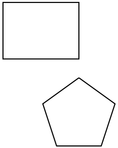
Is $\mathrm{ex}(n,\{K_4^-,L_2\}) = \frac{1}{4}\binom{n}{3}(1+o(1))$? $L_2=\{123,124,125,136,456\}$

Thank you for your attention!

All triangles?



Other Shapes?



in $\mathbb{R}^d$?