Flag Algebras and Its Application

Bernard Lidický

CombinaTexas 2023
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Outline

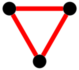
- Flag Algebras Introduction
- "Proof" of Mantel's theorem
- Erdős Pentagon Problem and inducibility
- Crossing numbers
- ℓ_2-norm in Turán type problems
- ε-similar triangles
- Small Ramsey numbers

We will not make it all the way....

Inspirational Problem

- Let n be fixed number of vertices in a graph G.
- Assume G has m edges.
- What is the number of triangles in G?

edge is 

triangle is 

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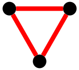
triangle is 

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Liu, Pikhurko, Staden 2020 (144 pages)

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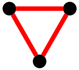
triangle is 

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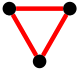
triangle is 

Consider $n \to \infty$.
$\#$ Edges $= p\binom{n}{2}$
$\#$ Triangles $= t\binom{n}{3}$
Now $p, t \in [0, 1]$.

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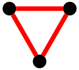
triangle is 

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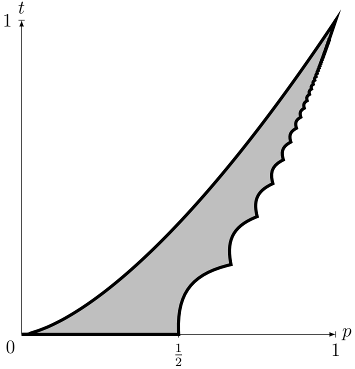

Upper bound $p^{3/2}$ Kruskal-Katona 1964
Asymptotic lower bound by Razborov 2008

Flag algebras

Seminal paper:
Razborov, Flag Algebras, *Journal of Symbolic Logic* **72** (2007), 1239–1282.
David P. Robbins Prize by AMS for Razborov in 2013 over 300 citations (on google)

Example

If density of edges is p, what is the minimum density of triangles?

- Designed to attack extremal problems.
- Works well if constraints as well as desired value can be computed by checking small subgraphs (or average over small subgraphs).
- The results are for the limit as graphs get very large.

Example extremal problem

Theorem (Mantel 1907)

Every n-vertex triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

Problem

Maximize a graph parameter (# of edges) over a class of graphs (triangle-free).

- local condition and global parameter (computable locally)
- threshold
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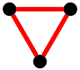
We will use colors for edges and non-edges.

Flag algebras definitions

Let G be a 2-edge-colored complete graph on n vertices.



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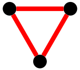

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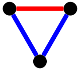

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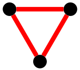

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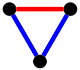

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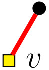

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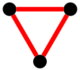

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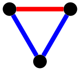

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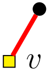

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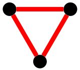

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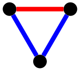

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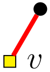

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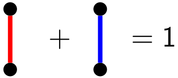

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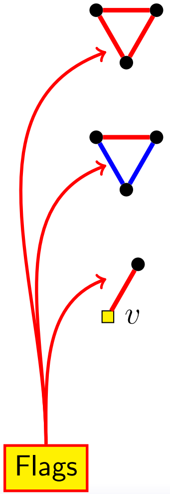

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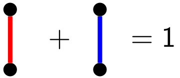

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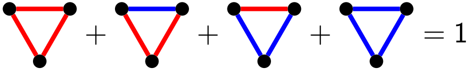
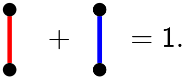


Same kind as



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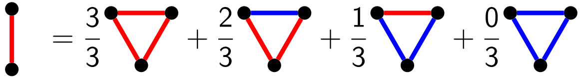

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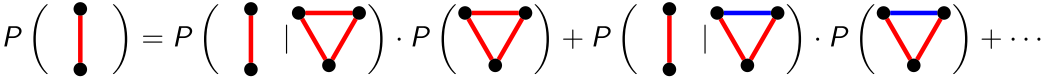

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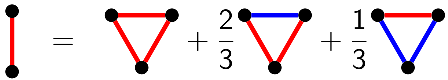

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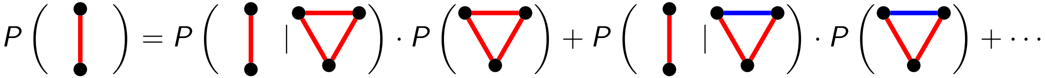

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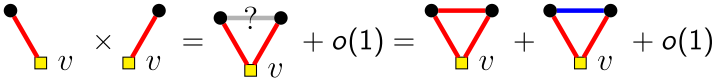

$o(1)$ as $n \to \infty$ (will be omitted on next slides)

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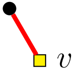

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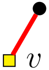

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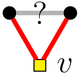

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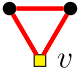

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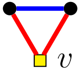

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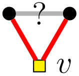

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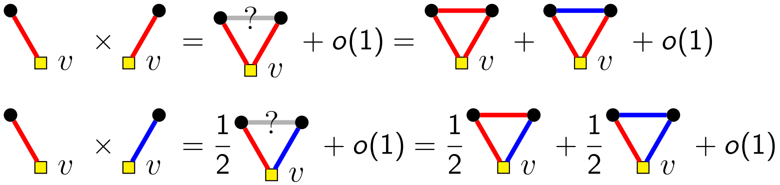

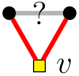

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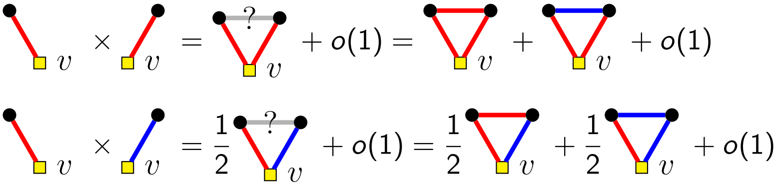

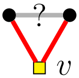

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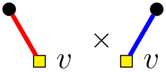

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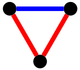
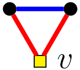
$\frac{1}{3}$  $= \frac{1}{n} \sum_{v \in V(G)}$  v

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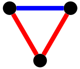
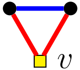
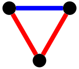
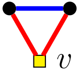
$\frac{1}{3}$  $= \frac{1}{n} \sum_{v \in V(G)}$  v

 $\binom{n}{3} = \sum_{v \in V(G)}$  $v \binom{n-1}{2}$

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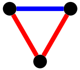
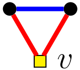
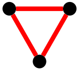
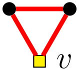
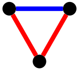
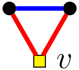
$\frac{1}{3}$  $= \frac{1}{n} \sum_{v \in V(G)}$  v

 $= \frac{1}{n} \sum_{v \in V(G)}$  v

 $\binom{n}{3} = \sum_{v \in V(G)}$  v $\binom{n-1}{2}$

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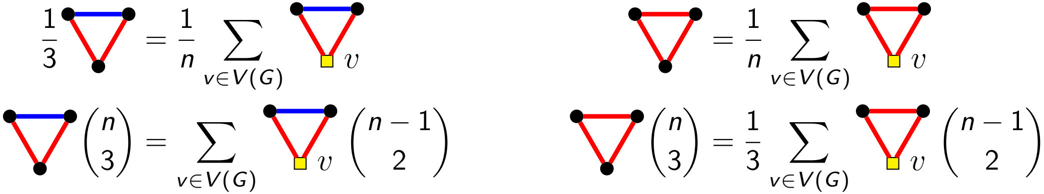
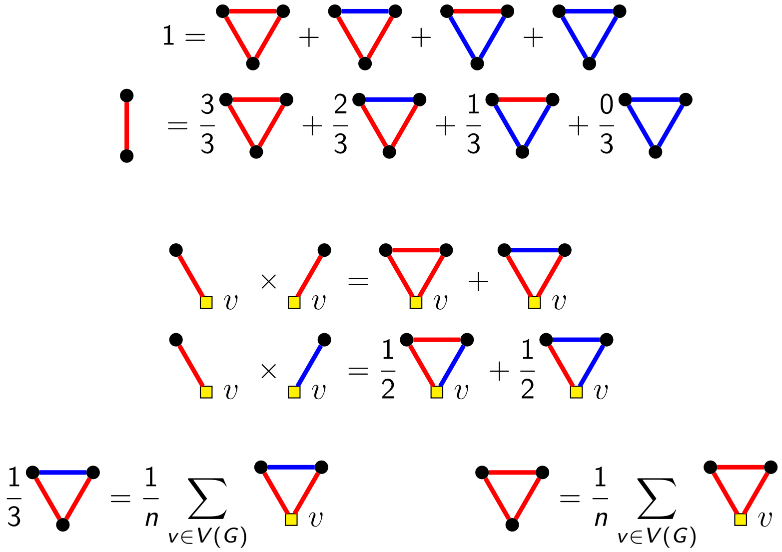


Identities Summary



Example - Mantel's Theorem

Theorem (Mantel 1907)

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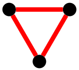
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

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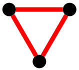
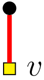
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$0 \leq \Big(1 - 2$  $\Big)^2$

Example - Mantel's Theorem

Theorem (Mantel 1907)

A triangle-free n-vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2}\binom{n}{2}$ *edges.*

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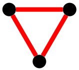
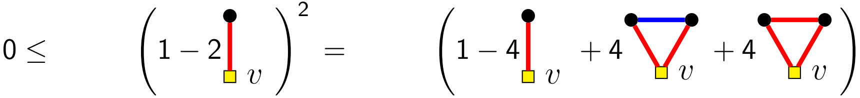
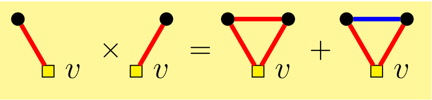
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)





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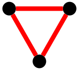
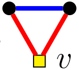
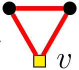
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$0 \leq \frac{1}{n}\sum_{v}\Big(1 - 2$  $v\Big)^2 = \frac{1}{n}\sum_{v}\Big(1 - 4$  $v + 4$  $v + 4$  $v\Big)$

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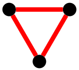
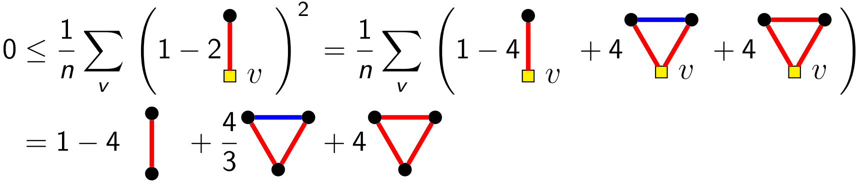
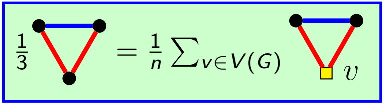
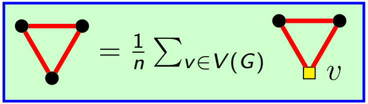
Assume  $= 0$. (We want to conclude  $\le \frac{1}{2}$.)







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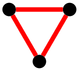
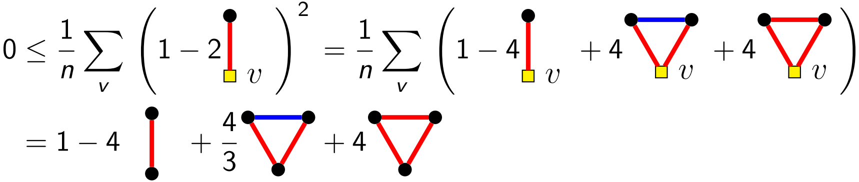
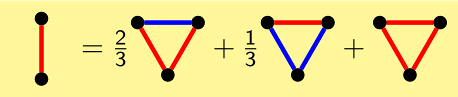
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)





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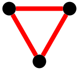
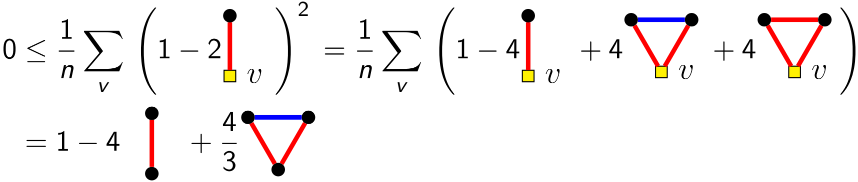
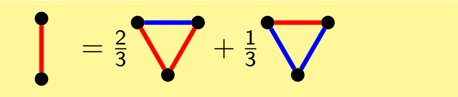
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)





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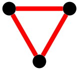
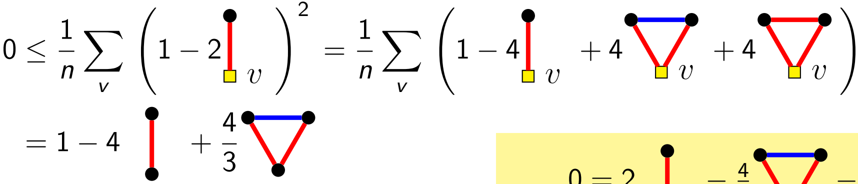
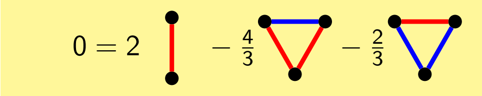
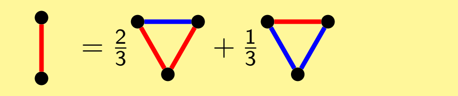
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)







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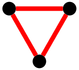
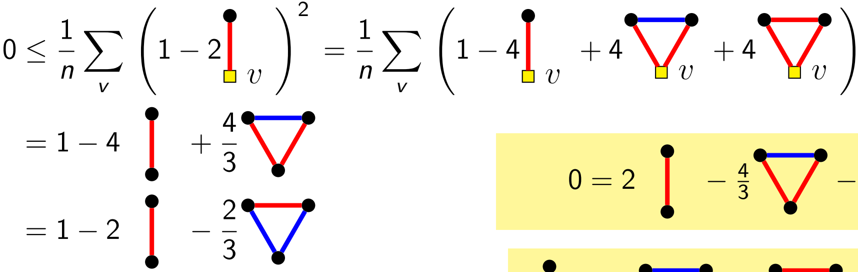
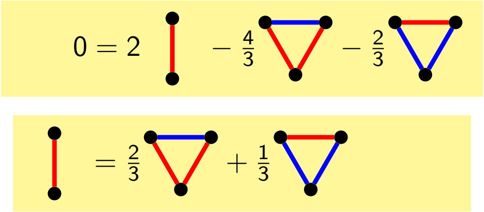
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)





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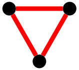
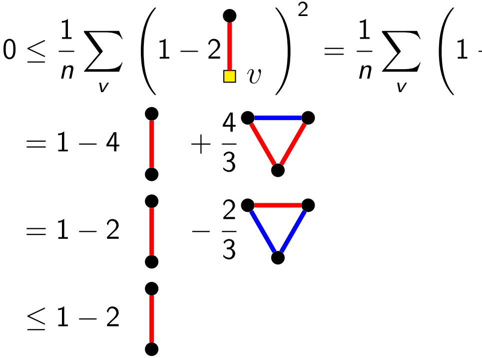
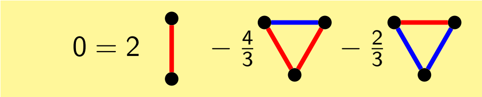
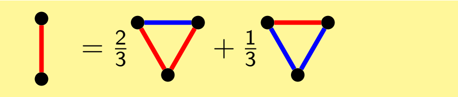
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)



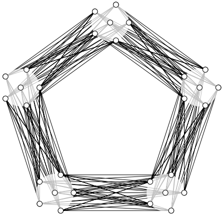


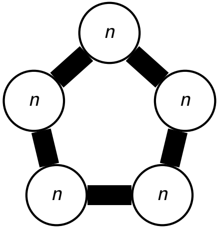


Erdős Pentagon Problem

Problem (Erdős, 83)

Is it true that a triangle-free graph on $5n$ vertices can contain at most n^5 pentagons?





Pentagons in triangle-free graphs

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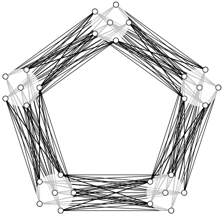

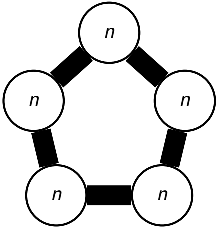

Theorem (Grzesik '12 & Hatami, Hladký, Kráľ, Norin, Razborov '13)

For all $n > n_0$ or $5|n$, the balanced blow-up of C_5 maximizes the number of C_5s over all triangle free graphs, and it is unique.

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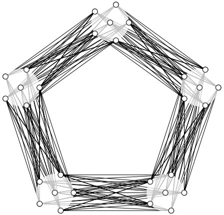

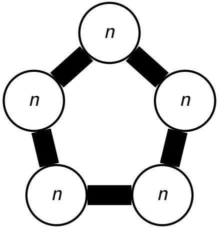

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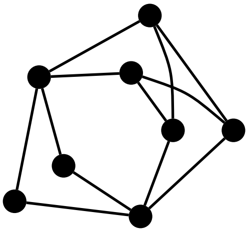
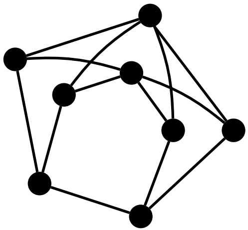
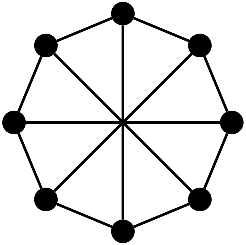
For all n ~~$> n_0$ or $5|n$~~, the balanced blow-up of C_5 maximizes the number of C_5s over all triangle free graphs, and it is unique unless $n < 5$ or $n = 8$.

Extremal examples on 8 vertices

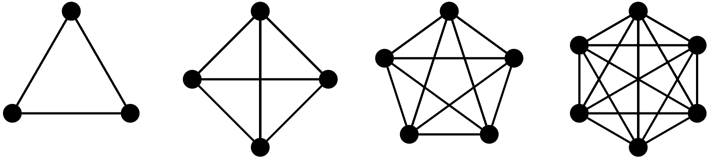






Problem (Palmer, 2018)

Which K_r-free graph on n vertices contains the most pentagons?



K_3 K_4 K_5 K_6

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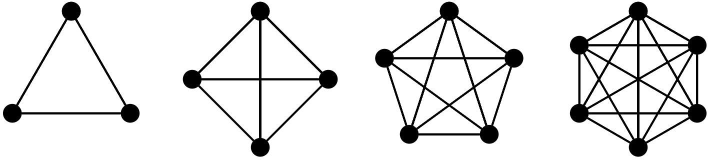

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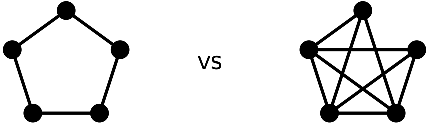
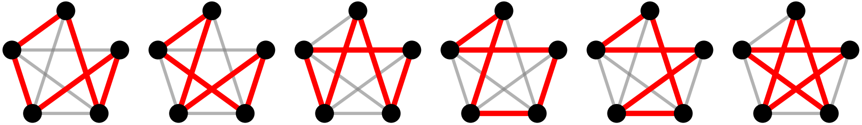
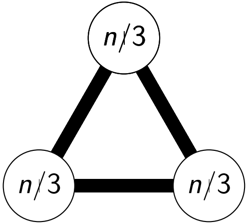




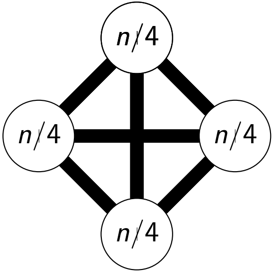
Every C_5 counts

Theorem (L., Murphy (2021))

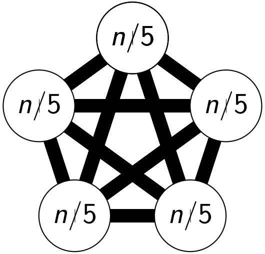
For all $r \geq 3$, the number of 5-cycles among K_{r+1}-free graphs is maximized by the Turán's graph $T_r(n)$ for n sufficiently large.



$T_3(n)$



$T_4(n)$



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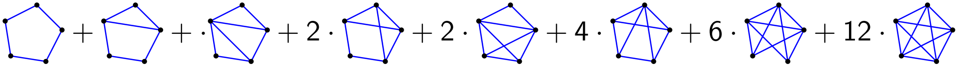
Flag Algebras formulation:

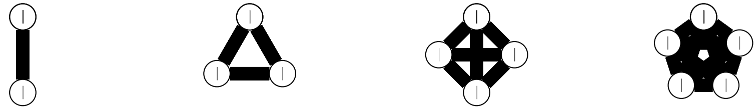
Maximize



Subject to $K_{r+1} = 0$

Maximizing Other Graphs in K_r-free

$\text{ex}(n, \#H, F) :=$ Maximum number of copies of H in F-free graph on n vertices.



$T_2(n)$ $T_3(n)$ $T_4(n)$ $T_5(n)$

Theorem (Mantel (1907))

$ex(n, \#K_2, K_3) = |E(T_2(n))|$. Moreover, $T_2(n)$ is the unique extremal graph.

Theorem (Turán (1941))

$ex(n, \#K_2, K_{r+1}) = |E(T_r(n))|$ for $r \geq 3$, and $T_r(n)$ is the unique extremal graph.

Theorem (Erdős-Stone (1946), Erdős-Simonovits (1966))

$$ex(n, \#K_2, F) = \left(1 - \frac{1}{\chi(F)-1}\right)\frac{n^2}{2} + o(n^2).$$

Maximizing Other Graphs in K_r-free

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$T_2(n)$ $T_3(n)$ $T_4(n)$ $T_5(n)$

Theorem (Zykov (1949))

Let $t \le r$. $ex(n, \#K_t, K_{r+1})$ is maximized in $T_r(n)$.

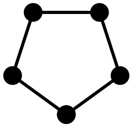
Theorem (Alon, Shikhelman (2015))

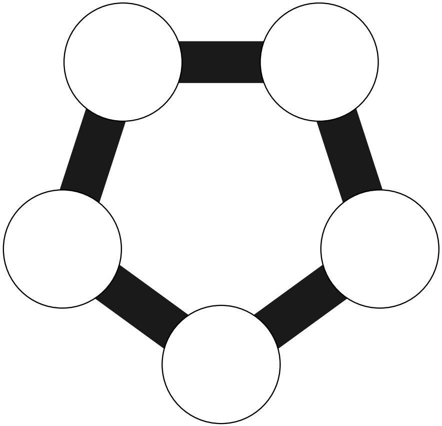
$ex(n, \#K_3, C_5) \le (1 + o(1))\frac{\sqrt{3}}{2} n^{3/2}$

Recent results by Gerbner+Palmer, Ma+Qui, Qian+Xie+Ge, Murphy+Nir

Maximizing Pentagons (Induced Version)

Maximize





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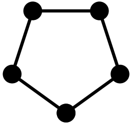

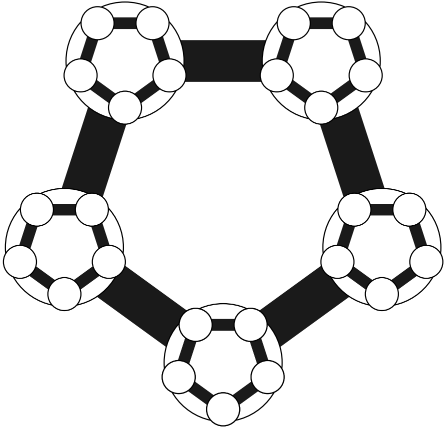

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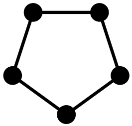
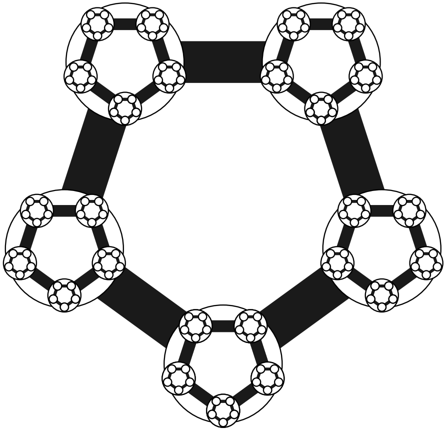
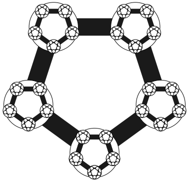




Results

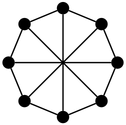
Theorem (Balogh, Hu, L., Pfender, 2016)

The iterated blow-up of C_5 maximizes the number of 5-cycles on 5^n vertices.

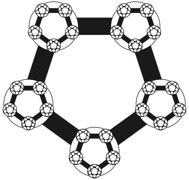


Theorem (L., Mattes, Pfender, 2023)

The iterated blow-up of C_5 maximizes the number of 5-cycles on n vertices. Except $n = 8$.



Fractalizers



A graph G is a *fractalizer* if the graph maximizing the number of induced copies of G is an iterated blow-up of G.

Theorem (Fox, Huang, Lee 2015+, Yuster 2019)

Almost every graph is a fractalizer.

Fractalizers

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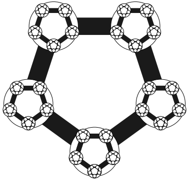

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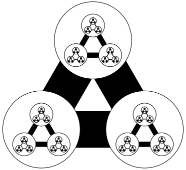
Almost every graph is a fractalizer.

Can you find some? Other than K_n or $\overline{K_n}$.



Conjecture (Pippinger, Golumbic 1975)

Cycles, except 4-cycle, are fractalizers.

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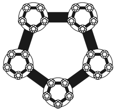
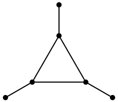


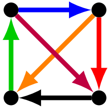
Theorem (L., Mattess, Pfender 2023)

5*-cycle is almost a fractalizer; exception on 8 vertices.*



Theorem (Blumenthal, Phillips, 2021+)

Net is a fractalizer if on 6^k *vertices.*



Theorem (Mubayi, Razborov 2021)

Rainbow tournaments on at least 4 vertices are fractalizers.

Graph fractalizers need at least 8 vertices.

Crossing numbers

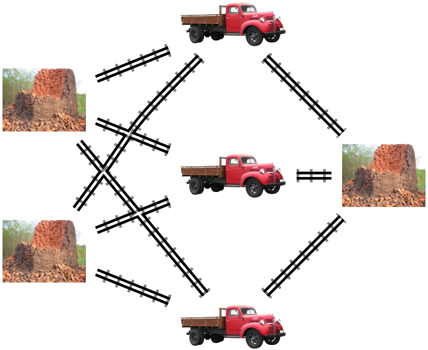
Application to graph drawing.

Crossing number

Turán 1945: In a forced labor camp, prisoners transfer carts of bricks from kilns to shipping yards.

When two tracks cross, cart is likely to derail.

How to connect every kiln and shipping yard that minimizes the number of crossings?



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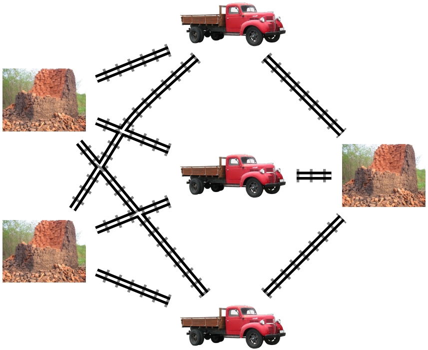
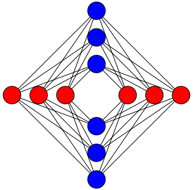


$K_{m,n}$ is a complete bipartite graph with sizes m and n, $K_{3,3}$ is above.

For a graph G, $cr(G)$ is the *crossing number*.

Conjecture (Zarankiewicz 1954)

$$cr(K_{m,n}) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{(n-1)}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{(m-1)}{2} \right\rfloor$$



$K_{6,6}$

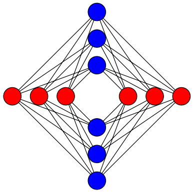
Theorem (Norin, Zwols 2013)

$$cr(K_{m,n}) \geq 0.905 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{(n-1)}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{(m-1)}{2} \right\rfloor$$

for large m and n. (Zarankiewicz's conjecture is 90.5% true)

80% Kleitman 1970
83% De Klerk, Maharry, Pasechnik, Richter, Salazar 2006
85.9% De Klerk, Pasechnik, Schrijver 2007
88.7% Brosch, Polak 2022+

Rectilinear crossing number is with straight line drawing.



$K_{6,6}$

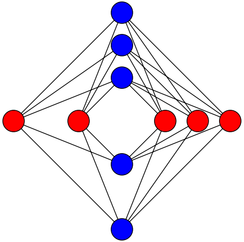
Theorem (Balogh, L., Norin, Pfender, Salazar, Spiro 2023+)

Zarankiewicz's conjecture is 91.1% true for large m and n.

Theorem (Balogh, L., Norin, Pfender, Salazar, Spiro 2023+)

Rectilinear version of the Zarankiewicz is 97.3% true for large m and n.

$K_{5,5,5}$



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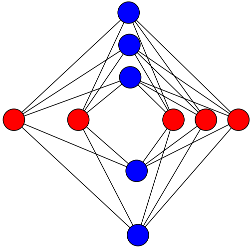

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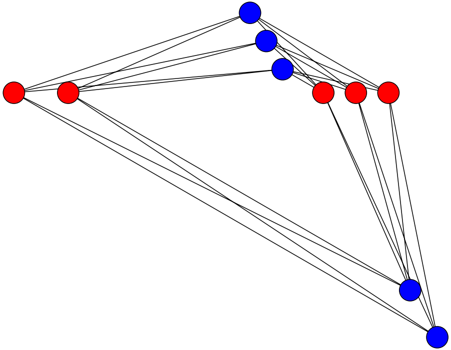
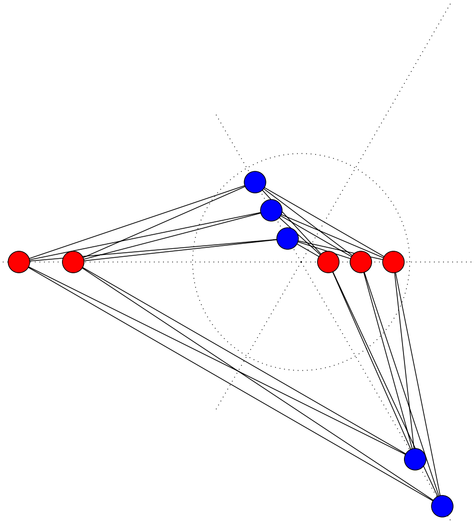


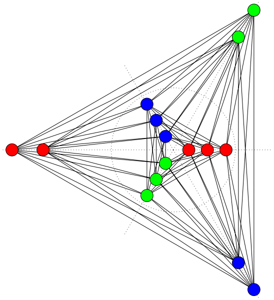
$K_{5,5,5}$



For a graph G, $\overline{cr}(G)$ is the rectilinear crossing number.

Conjecture

$\overline{cr}(K_{n_1,n_2,n_3})$ *is minimized by*



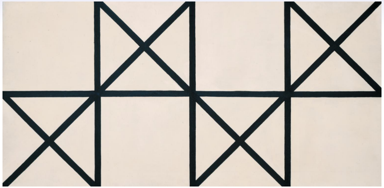
$K_{5,5,5}$

Theorem (Gethner, Hogben, L., Pfender, Ruiz, Young, '17)

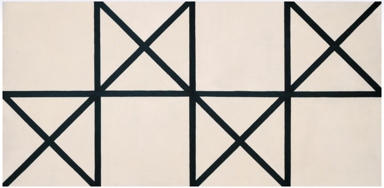
$\overline{cr}(K_{n_1,n_2,n_3})$ *conjecture is 89% true for large* n_1, n_2, *and* n_3.

Problem

What about partite graphs with more parts?



Anthony Hill
Orthogonal / Diagonal
Composition
1954



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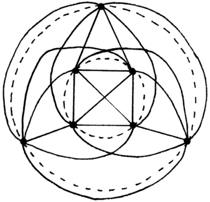
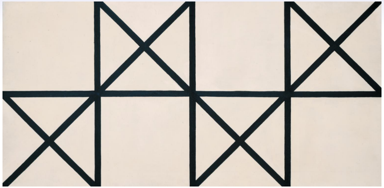


Hill considered crossing number of complete graphs.



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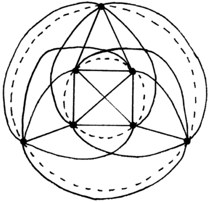

Hill considered crossing number of complete graphs.

Conjecture (Hill 1962)

$cr(K_n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$

Conjecture (Hill 1962)

$$cr(K_n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

Conjecture is true

- if $n \leq 12$.
- 100% with various additional restrictions on the drawing
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Conjecture (Hill 1962)

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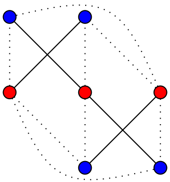
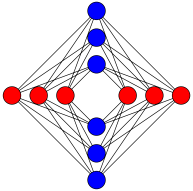
Theorem (Balogh, L., Salazar 2019)

Conjecture is 98.5% true.

Tuán type result





$K_{6,6}$

Theorem (Balogh, L., Norin, Pfender, Salazar, Spiro 2023+)

If $K^D_{n,n}$ is a drawing of $K_{n,n}$ where no $K_{3,4}$ induces exactly two crossings sharing one vertex, then $K^D_{n,n}$ has at least $n^4/16 + o(n^4)$ crossings. (100% True)

How is it done?

For multipartite graphs

- Color vertices to indicate parts.
- For every $\{(a, b), (c, d)\}$, where a, b, c, d are vertices remember if edges ab and cd cross or not.
- Necessary to generate all (combinatorial) embeddings of graphs on n vertices.

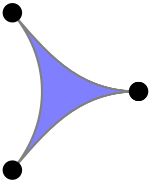
For complete graphs

- For every vertex remember clockwise order of its neighbors.
- Necessary to generate all (combinatorial) embeddings of graphs on n vertices.

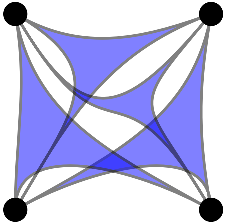
ℓ_2-norm in Turán Type Problems

Hypergraph Setting

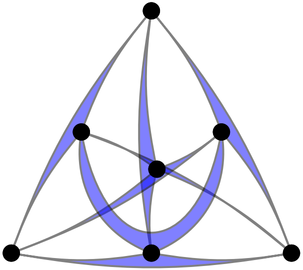
3-uniform hypergraphs have triples of vertices as edges.



One Edge, K_3^3



Complete hypergraph K_4^3



Fanno Plane

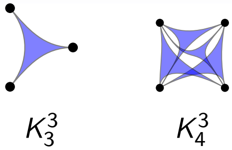
Turán's Tetrahedron problem: Determine $\mathrm{ex}(n, \# K_3^3, K_4^3)$

\$500 reward by Erdős

Turán's Tetrahedron problem

Determine $\mathrm{ex}(n, \#K_3^3, K_4^3)$
Asymptotic setting:

$$\pi(K_4^3) = \lim_{n\to\infty} \mathrm{ex}(n, \#K_3^3, K_4^3)/\binom{n}{3}$$



Theorem (Kostochka 1982, Brown 1983, Fon-der-Flaass 1988, Frohmade 2008)

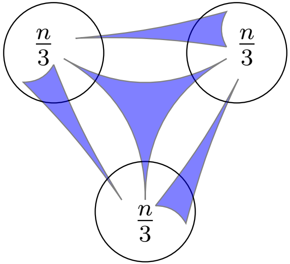
$\pi(K_4^3) \geq 5/9$

Theorem (Baber 2012)

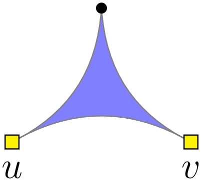
$\pi(K_4^3) \leq 0.5615$

Theorem (Razborov 2010)

$\pi(K_4^3, \textit{few other graphs}) = 5/9$



Co-degree vector



Let G be an n-vertex 3-uniform hypergraph
co-degree vector $X \in \mathbb{Z}^{\binom{n}{2}}$ is indexed by pairs $u, v \in V(G)$
$X_{u,v} := \#$ edges containing u and v.

$$\#\text{edges} = \frac{1}{3}\sum_{uv} X_{uv} = \frac{1}{3}||X||_1$$

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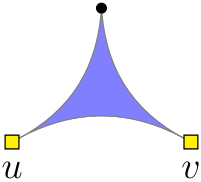
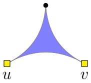
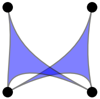
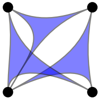
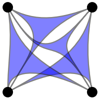


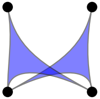
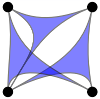
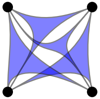
$$\#\text{edges} = \frac{1}{3}\sum_{uv} X_{uv} = \frac{1}{3}||X||_1$$

New idea: Consider

$$\sum_{uv} X_{uv}^2 = (||X||_2)^2$$

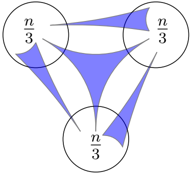
$\frac{1}{\binom{n}{2}} \sum_{uv}$  ^{2} $= \frac{1}{6}$  $+ \frac{1}{2}$  $+$ 

Let $\mathcal{G}_n$ be H-free 3-uniform hypergraphs on n vertices.

$$\sigma(H) := \lim_{n\to\infty} \max_{G\in\mathcal{G}_n} \frac{1}{6}$$  $$+ \frac{1}{2}$$  $$+$$ 

Theorem (Balogh, Clemen, L. 2022)

$\sigma(K_4^3) = \frac{1}{3}$



Theorem (Balogh, Clemen, L. 2022)

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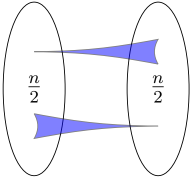

Any many others.

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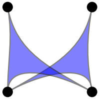
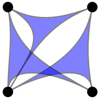
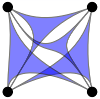
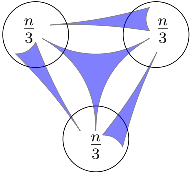
$\sigma(H) := \lim_{n\to\infty} \max_{G\in\mathcal{G}_n} \frac{1}{6}$  $+ \frac{1}{2}$  $+$ 



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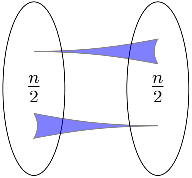

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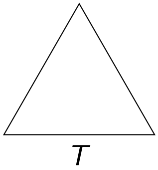
Any many others.

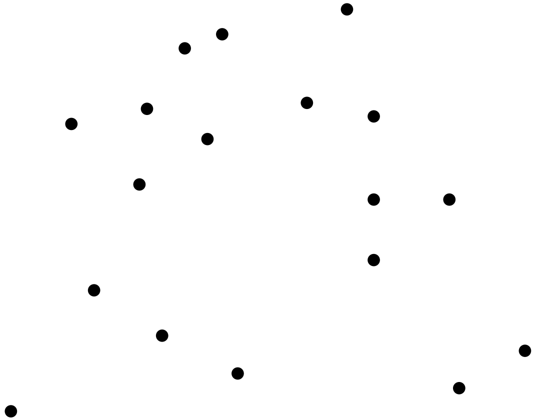
Future: Better exactness methods and other settings applications.

ε-similar Triangles

Problem

Let T be a triangle and $n \in \mathbb{N}$ fixed.
Which n points in $\mathbb{R}^2$ maximize the number of triangles similar to T?





T_1 and T_2 are *ε-similar* if their inner angles differ by at most ε.
(OK to move, scale, rotate, ε-perturb)

PROBLEM

Let T be a triangle and $n \in \mathbb{N}$ fixed. (and $\varepsilon > 0$ fixed)
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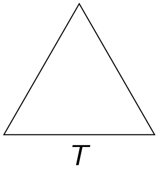

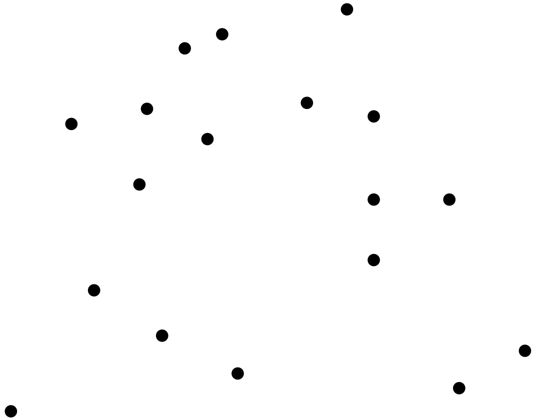

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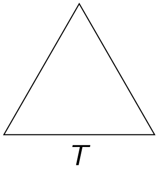

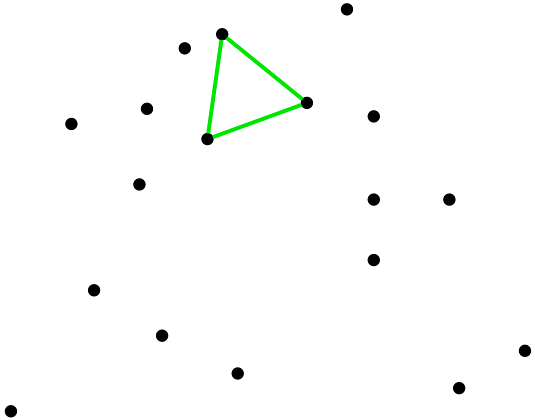

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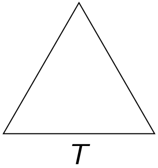

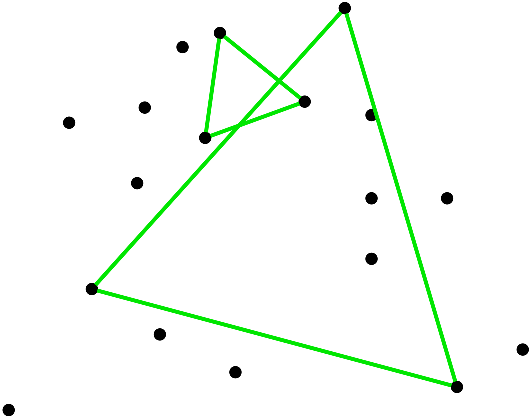

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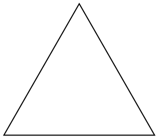
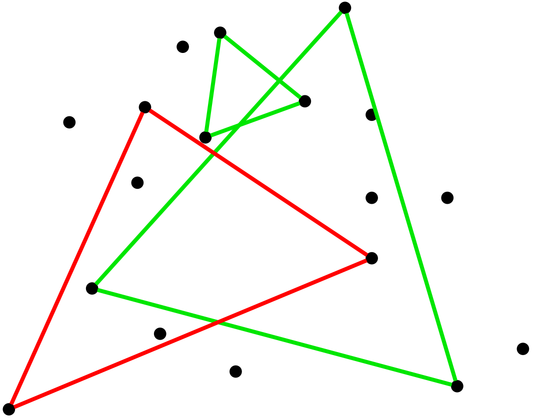
Problem

Let T be a triangle and $n \in \mathbb{N}$ fixed. (and $\varepsilon > 0$ fixed)
Which n points in $\mathbb{R}^2$ maximize the number of triangles similar to T?



T



T_1 and T_2 are *ε-similar* if their inner angles differ by at most ε.
(OK to move, scale, rotate, ε-perturb)

Lower bound construction

Let T be a triangle and $n \in \mathbb{N}$ fixed. (and $\varepsilon > 0$ fixed)
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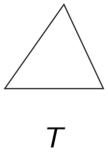

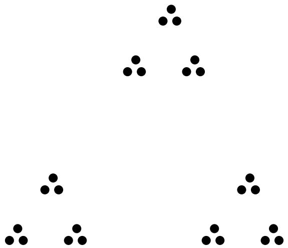

$h(n, T, \varepsilon) :=$ max # of *ε-similar* triangles to T, it is at least $\frac{1}{4}\binom{n}{3}(1 + o(1))$.

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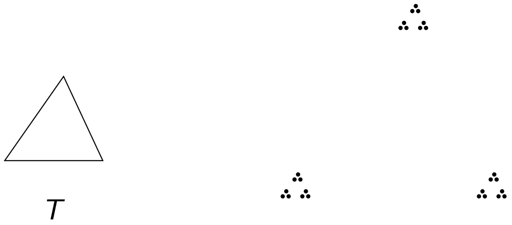

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Results

Theorem (Bárány and Füredi (2019))

For almost every triangle T there is an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$,

$$h(n, T, \varepsilon) \leq 0.25072\binom{n}{3}(1 + o(1)).$$

If T is equilateral, then $h(n, T, \varepsilon) = \frac{1}{4}\binom{n}{3}(1 + o(1))$

Theorem (Balogh, Clemen, L. (2022))

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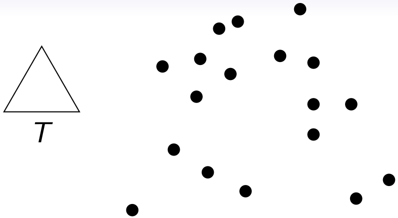
$h(n, T, \varepsilon) :=$ max # of *ε-similar* triangles to T, it is at least $\frac{1}{4}\binom{n}{3}(1 + o(1))$.

Let T and ε are given

- Fix n points in the plane.
- For every T' ε-similar to T, add a 3-edge
- Investigate the resulting hypergraph H

H has no subhypergaph in $\mathcal{F} = \{K_4^3, \dots\}$



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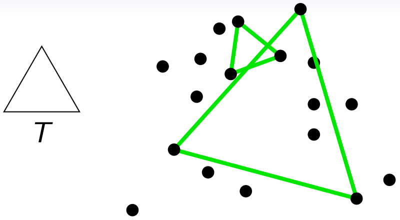

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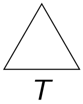

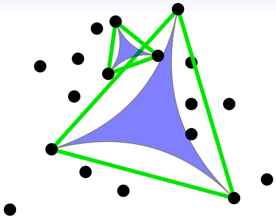

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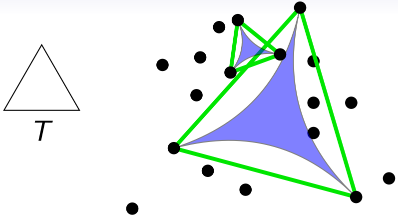

Theorem (Balogh, Clemen, L. (2022))

$\mathcal{F}$-free hypergraph has at most $\frac{1}{4}\binom{n}{3}(1+o(1))$ edges.

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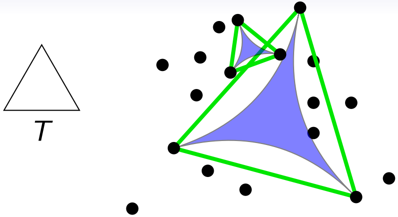

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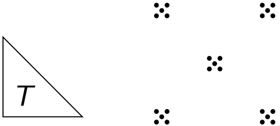
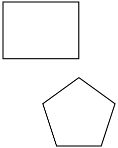
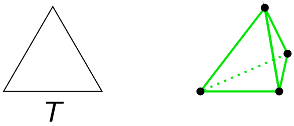
All triangles?



Other Shapes?



in $\mathbb{R}^d$?

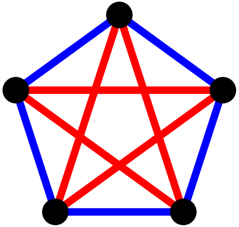


Small Ramsey numbers

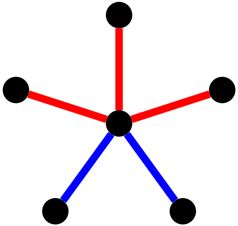
Definition

$R(G_1, G_2, \ldots, G_k)$ is the smallest integer n such that any k-edge coloring of K_n contains a copy of G_i in color i for some $1 \le i \le k$.

$R(K_3, K_3) > 5$



$R(K_3, K_3) \le 6$



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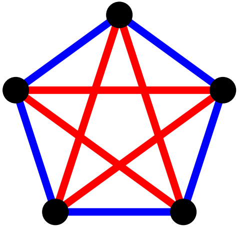

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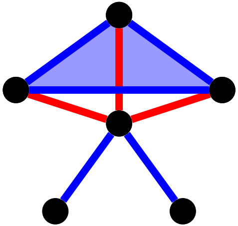

Theorem (Ramsey 1930)

$R(K_m, K_n)$ is finite.

$R(G_1, \ldots, G_k)$ is finite

Questions:

- study how $R(G_1, \ldots, G_k)$ grows if $G_1, \ldots, G_k$ grow (large)
- study $R(G_1, \ldots, G_k)$ for fixed $G_1, \ldots, G_k$ (small)

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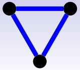
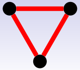
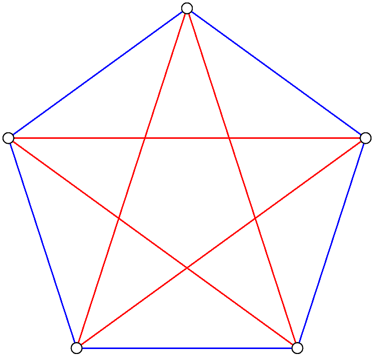
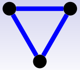
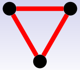
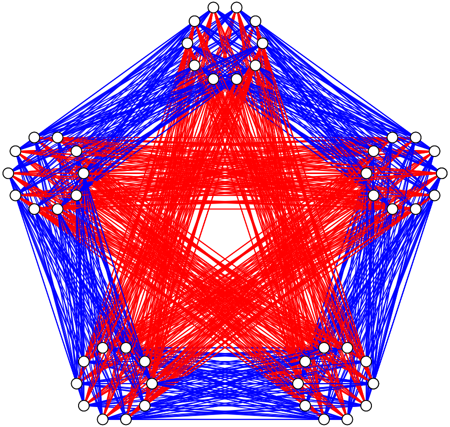
Radziszowski - *Small Ramsey Numbers*
Electronic Journal of Combinatorics - Survey

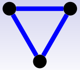
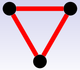
[Erdős] Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack.

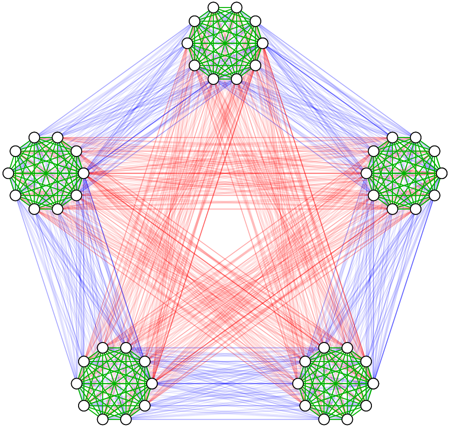
Take any graph G with no  and  .

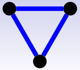
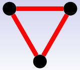


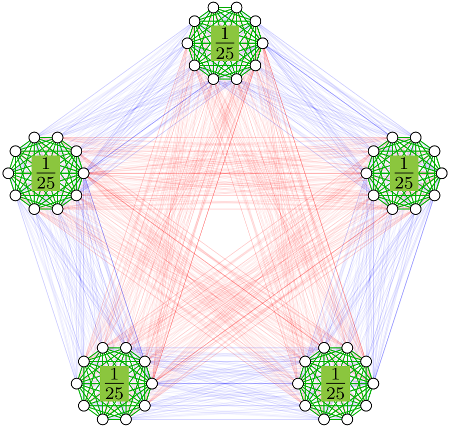
Take any graph G with no  and  .



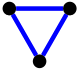
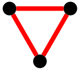
Take any graph G with no  and  .



Take any graph G with no  and .



If G has k vertices, then the blow-up has density of non-edges $\geq \frac{1}{k}$.

Take any graph G with no  and  .

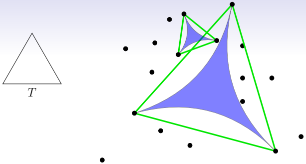
If G has k vertices, then the blow-up has density of non-edges $\geq \frac{1}{k}$.
If any blow-up has density of non-edges $\geq \frac{1}{k}$ then G has $\leq k$ vertices.

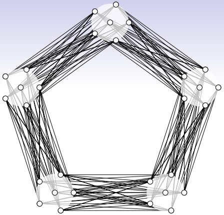
New upper bounds (L., Pfender 2021)

Problem	Lower	New upper	Old upper
$R(K_4^-, K_8^-)$	29	32	38
$R(K_4^-, K_9^-)$	31	46	53
$R(K_4, K_7^-)$	37	49	52
$R(K_5^-, K_6^-)$	31	38	39
$R(K_5^-, K_7^-)$	40	65	66
$R(K_5, K_6^-)$	43	62	66
$R(K_5, K_7^-)$	58	102	110
$R(K_6^-, K_7^-)$	59	124	135
$R(K_7, K_4^-)$	28	29	30
$R(K_8, K_4^-)$	29	39	42
$R(K_8, C_5)$	29	29	33
$R(K_9, C_5)$	33	36	
$R(K_9, C_6)$	41	41	
$R(K_9, C_7)$	49	58	
$R(K_{2,2,2}, K_{2,2,2})$	30	32	60?

Problem	Lower	New upper	Old upper
$R(K_{3,4}, K_{2,5})$		20	21
$R(K_{3,4}, K_{3,3})$		20	25
$R(K_{3,4}, K_{3,4})$		25	30
$R(K_{3,5}, K_{1,6})$	17	17	
$R(K_{3,5}, K_{2,4})$	16	20	
$R(K_{3,5}, K_{2,5})$	21	23	
$R(K_{3,5}, K_{3,3})$		24	28
$R(K_{3,5}, K_{3,4})$		29	33
$R(K_{3,5}, K_{3,5})$	30	33	38
$R(K_{4,4}, K_{4,4})$	30	49	62
$R(W_7, W_4)$		21	
$R(W_7, W_5)$		16	
$R(W_7, W_6)$		19	
$R(B_4, B_5)$	17	19	20
$R(B_3, B_6)$	17	19	22
$R(B_5, B_6)$	22	24	26

Problem	Lower	New upper	Old upper
$R(W_5, K_6)$	33	36	
$R(W_5, K_7)$	43	50	
$R(Q_3, Q_3)$	13	13	14
$R(K_3, C_5, C_5)$	17	17	21?
$R(K_3, C_4, C_4, C_4)$	24	29	
$R(K_4, C_4, C_4)$	52	71	72
$R(K_4^-, K_4^-, K_4^-)$	28	28	30
$R(K_3, K_4^-, K_4^-)$	21	23	27
$R(K_4, K_4^-, K_4^-)$	33	47	59
$R(K_4, K_4, K_4^-)$	55	104	113
$R(K_3, K_4, K_4^-)$	30	40	41
$R(K_4^-, K_5^-; 3)$	12	12	
$R(K_4^-, K_5; 3)$	14	16	
$R(K_4^-, K_4^-, K_4^-; 3)$	13	14	16





Thank you for your attention!