Flag Algebras and Weighted Turán problems with applications to Ramsey-Turán questions

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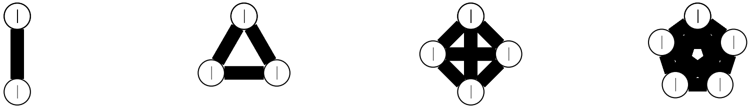
Ramsey and Turán

Theorem (Ramsey (1930))

For every r, s exists $R(r,s)$ such that every graph on $R(r,s)$ vertices contains K_r or $\overline{K_s}$.

Theorem (Turán (1941))

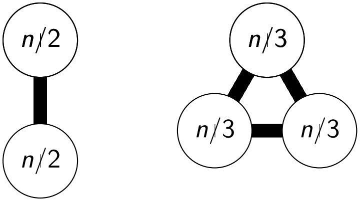
K_q-free graph on n vertices maximizing the number of edges is $T_{q-1}(n)$.



$T_2(n)$ $T_3(n)$ $T_4(n)$ $T_5(n)$

Problem

What K_q-free graph on n vertices maximizing the number of edges while having low independence number?



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What K_q-free graph on n vertices is maximizing the number of edges while having low p-independence number?

p-independence number of a graph G is

$$\alpha_p(G) := \max\{|U| : U \subseteq V(G) \text{ and } G[U] \text{ is } K_p\text{-free}\}$$

Note $\alpha_2(G) = \alpha(G)$

Ramsey-Turán number

$$RT_p(n, K_q, m) := max\{e(G) : G \text{ is } K_q\text{-free}, v(G) = n, \alpha_p(G) \leq m, \}$$

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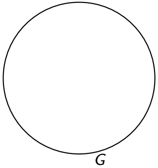

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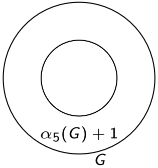

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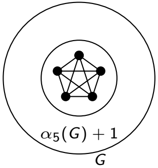

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Asymptotic version

$$\varrho_p(q) := \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{RT_p(n, K_q, \varepsilon n)}{\binom{n}{2}}$$

Conjecture (Erdős, Hajnal, Simonovits, Sós and Szemerédi '94)

The asymptotic extremal graph G for $\varrho_p(q)$ has the following structure. Let $q = pt + r + 2$, where $t \in \mathbb{N}$ and $r \in \mathbb{Z}_p$. Then there is a partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_t$ such that

- $e(G[V_i]) = o(n^2)$ for all $0 \leq i \leq t$;
- $d_G(V_0, V_1) = \frac{r+1}{p} - o(1)$, and degrees in $G[V_0, V_1]$ differ by $o(n)$;
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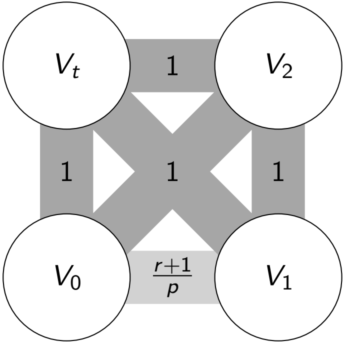

In particular
$\varrho_p(q) = \varrho_p^{\star}(q) := \frac{(t-1)(2p-r-1)+r+1}{t(2p-r-1)+r+1}$.

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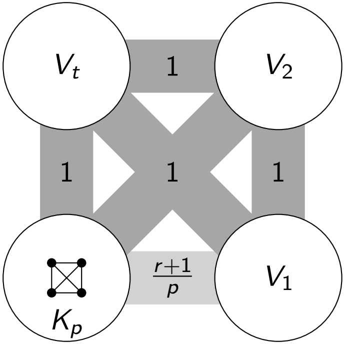

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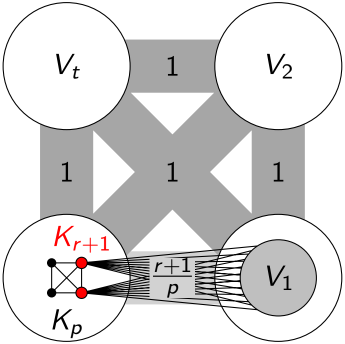

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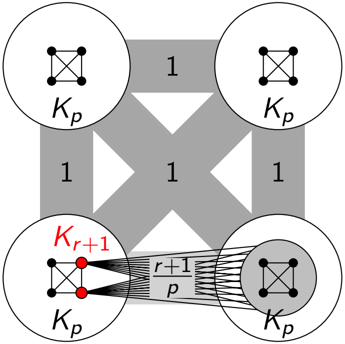

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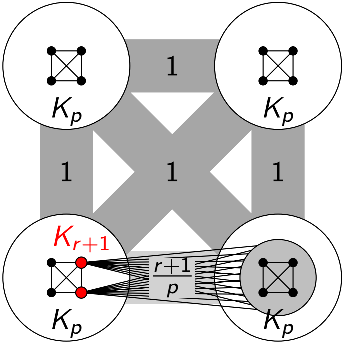

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Liu, Reiher, Sharifzadeh, and Staden $\varrho_{16}(22) = 1/6 > 5/32 = \varrho_{16}^\star(22)$

Conjectured construction

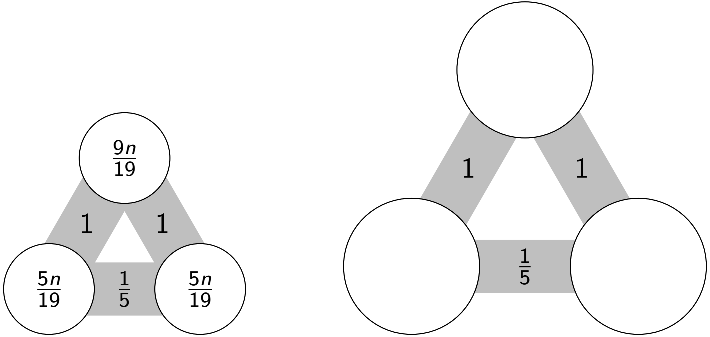

Figure: Sketch of a construction for $\varrho_5(12) \geq \frac{10}{19}$.

Liu, Reiher, Sharifzadeh, and Staden
Let $q = pt + \ell + 1$. Then for all $0 \leq \ell \leq p/2$: $\varrho_p(q) \geq \varrho_p^\star(q)$

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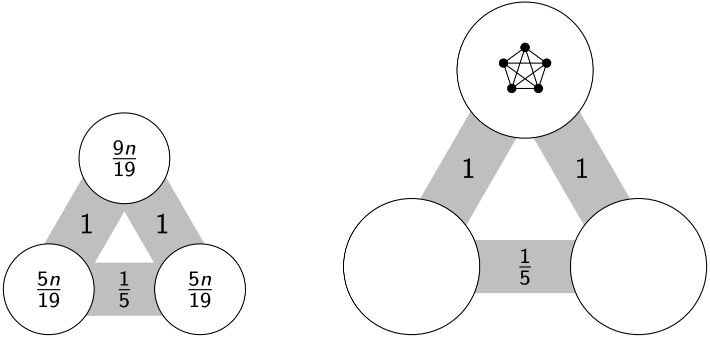

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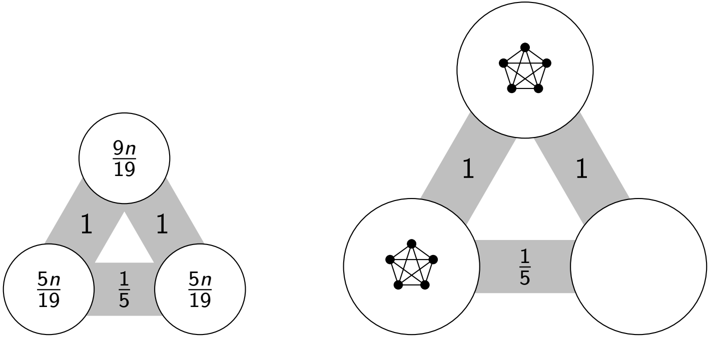

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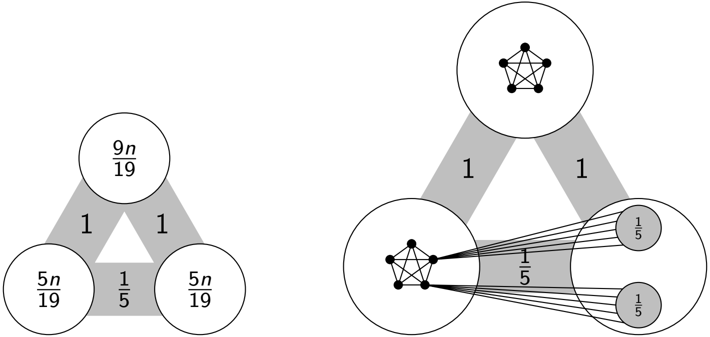

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Let $q = pt + \ell + 1$. Then for all $0 \leq \ell \leq p/2$: $\varrho_p(q) \geq \varrho_p^\star(q)$

Our work

We calculate upper bound on $\varrho_p(q)$ for some small values of p and q.

$$\varrho_p(q) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{RT_p(n, K_q, \varepsilon n)}{\binom{n}{2}}.$$

Plan

- Take large K_q-free n-vertex graph G where every εn vertices contain K_p.
- Apply Szemerédi Regularity Lemma
- Get reduced graph R
- Note R is edge-weighted graph
- Show R does not contain certain subgraphs
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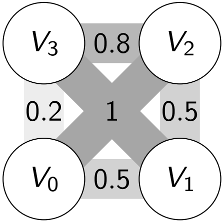
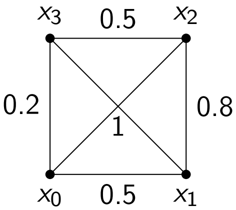


G



R

Weighted Turán Problems

An *edge weighting* w is $w : E(G) \to [0,1]$.

$$w(G) := \frac{2}{n^2} \sum_{e \in E(G)} w(e).$$

A *weighted clique* is (r, f)

$$f \colon \binom{[r]}{2} \to [0,1]$$

(G, w) *contains* (r, f) if exists injective

$$\phi \colon [r] \to V(G) \qquad \phi(i)\phi(j) \in E(G) \text{ and } w(\phi(i)\phi(j)) > f(ij)$$

Asymptotic problem

An *edge weighting* w is $w : E(G) \to [0,1]$.

$$w(G) := \frac{2}{n^2} \sum_{e \in E(G)} w(e).$$

Asymptotic Turán problem:

$$d(K_q) := \lim_{n\to\infty} \max_{|V(G)|=n,\ G \text{ is } K_q\text{-free}} e(G)/\binom{n}{2}$$

Set of weighted cliques $\mathcal{F}$

$$d(\mathcal{F}) := \lim_{n\to\infty} \max_{|V(G)|=n,\ G \text{ is } \mathcal{F}\text{-free}} w(G)$$

Weighted Turán

A *Turán edge weighting* $w_T : E(G) \to [0,1]$.

$$w_T(e) := \frac{r}{2(r-1)} \quad \text{where } r = \text{argmax}_k\{e \text{ is in } k\text{-clique in } G\}$$

$$w_T(G) := \frac{2}{n^2} \sum_{e \in E(G)} w_T(e).$$

Observation

For every $k \geq 2$

$$\lim_{n\to\infty} w_T(T_k(n)) = \frac{1}{2}$$

since $e(T_k(n)) = \frac{r-1}{r}\binom{n}{2}$.

Theorem (Bradač; Malec, Tompkins)

For every G *holds* $w_T(G) \leq \frac{1}{2}$.

Key lemma

$$g(A) := \max\left\{\mathbf{u}^\intercal A\mathbf{u} \,\middle|\, \mathbf{u} = (u_1, \ldots, u_m)^\intercal, \sum_{i=1}^{m} u_i = 1, u_i \geq 0\right\}.$$

A is *dense* if for every $i \in [m]$, $A_{i,i} = 0$ and A' obtained from A by removing i^{th} row and column satisfies $g(A') < g(A)$.

Lemma (Liu, Reiher, Sharifzadeh, and Staden 2021+)

Let $m \in \mathbb{N}$ and let $A = (a_{ij})$ be a dense symmetric $m \times m$ matrix with nonnegative entries and let $\mathbf{u}$ be optimal for A. Then

1. *A is positive, that is, $a_{ij} > 0$ for every $1 \leq i < j \leq m$,*
2. *$u_i > 0$ for every $i \in [m]$,*
3. *$\sum_{i\in[m]\setminus\{j\}} a_{ij}u_i = g(A)$, for every $j \in [m]$.*

Theorem (Bradač)

For every G holds $w_T(G) \leq \frac{1}{2}$.

Proof: Let $V(G) = v_1, \ldots, v_n$. Define $A \in \mathbb{R}^{n \times n}$

$$A_{i,j} = \begin{cases} w_T(v_i, v_j) & \text{if } (v_i, v_j) \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

for $\mathbf{x} = (1/n, \ldots, 1/n)$, we obtain

$$w_T(G) = \frac{2}{n^2} \sum_{e \in E(G)} w_T(e) = \mathbf{x}^\mathsf{T} A \mathbf{x} \leq g(A) \leq \frac{1}{2}$$

A' principal submatrix of A maximizing $g(A')$, pick minimal by inclusion
A' is dense, let $K \subseteq V(G)$ correspond to A'.
K induces a clique by Lemma
$a_{i,j} \leq w_T(|K|)$
$g(A) \leq g(A') \leq \sum_{i \in K} u_i \sum_{j \in K, j \neq i} u_j w_T(k) = w_T(k) \sum_{i \in K} u_i(1 - u_i) =$
$w_T(k) \left(1 - \sum_{i \in K} u_i^2\right) \leq w_T(k) \left(1 - \frac{1}{k}\right) = \frac{1}{2}$

Other weights

A *clique weighting* $cw : \mathbb{N} \to [0,1]$.

$$w(e) := cw(r) \quad \text{where } r = \text{argmax}_k \{e \text{ is in } k\text{-clique in } G\}$$

$$w(G) := \frac{2}{n^2} \sum_{e \in E(G)} w(e)$$

Theorem

Let cw be a clique weighting. Under mild assumptions, if $w(G)$ is close maximum, then G is close $T_r(n)$ for some r.

Other weights

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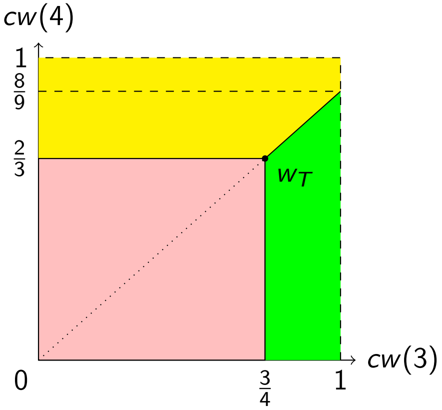
$w(G) := \frac{2}{n^2} \sum_{e \in E(G)} w(e)$

In K_5-free graphs, $cw(2) = 1$
If $cw(3) \leq 3/4$ and $cw(4) \leq 2/3$, then $T_2(n)$ is extremal.

If $cw(3) \geq 3/4$ and $cw(3) \geq \frac{9}{8} cw(4)$, then $T_3(n)$ is extremal.

If $cw(4) \geq 2/3$ and $cw(3) \leq \frac{9}{8} cw(4)$, then $T_4(n)$ is extremal.



Back to Ramsey-Turán

Ramsey-Turán number

$$RT_p(n, K_q, m) := max\{e(G) : G \text{ is } K_q\text{-free}, v(G) = n, \alpha_p(G) \leq m, \}$$

Asymptotic version

$$\varrho_p(q) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{RT_p(n, K_q, \varepsilon n)}{\binom{n}{2}}.$$

$\varrho_2(2t+1) = \frac{t-1}{t}$ for all $t \geq 1$, and $\varrho_2(2t) = \frac{3t-5}{3t-2}$ for all $t \geq 2$.

p, q	5	6	7	8	9	10	11	12	13	14
3	H	S	E	H		E	H		E	H
4	0	H	S	S	E	H	$\star$		E	H
5	0	0	S	S	S	S	E	$\star$		
6	0	0	0	S	S	S	S	$\star$	E	$\star$

Our addition

$$RT_p(n, K_q, m) := max\{e(G) : G \text{ is } K_q\text{-free}, v(G) = n, \alpha_p(G) \le m, \}$$

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$$\varrho_2(2t+1) = \frac{t-1}{t} \quad \text{for all} \quad t \ge 1, \qquad \text{and} \qquad \varrho_2(2t) = \frac{3t-5}{3t-2} \quad \text{for all } t \ge 2.$$

Theorem

The following bounds hold: $\varrho_4(11) \le \frac{4}{7}$, $\varrho_5(12) \le \frac{10}{19}$, $\varrho_6(12) \le \frac{5}{12}$, *and* $\varrho_6(14) \le \frac{12}{23}$.
In particular, $\varrho_5(12) = \frac{10}{19}$.

Translated to weighted Turán problems solved using flag algebras.

Proof sketch for $\varrho_5(12) \leq \frac{10}{19}$.

- Large K_{12}-free n-vertex graph G where every εn vertices contain K_5.
- Apply Szemerédi Regularity Lemma
- Get reduced graph R
- Note R is edge-weighted graph
- Show R does not contain certain subgraphs
- Compute an upper bound on edge density in R (Weighted Turán Problem)
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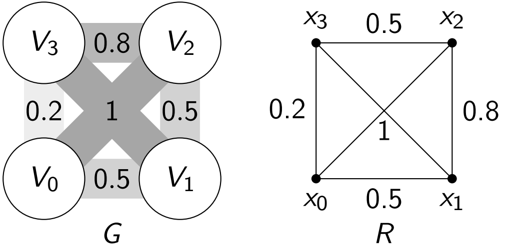
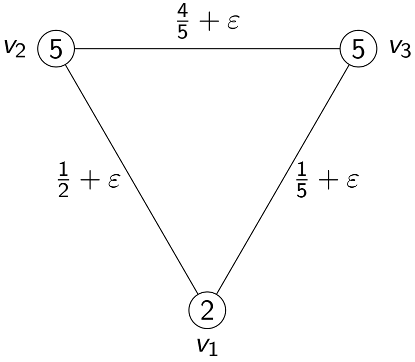
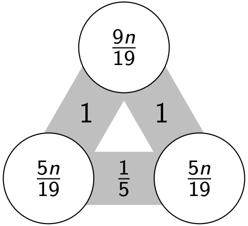


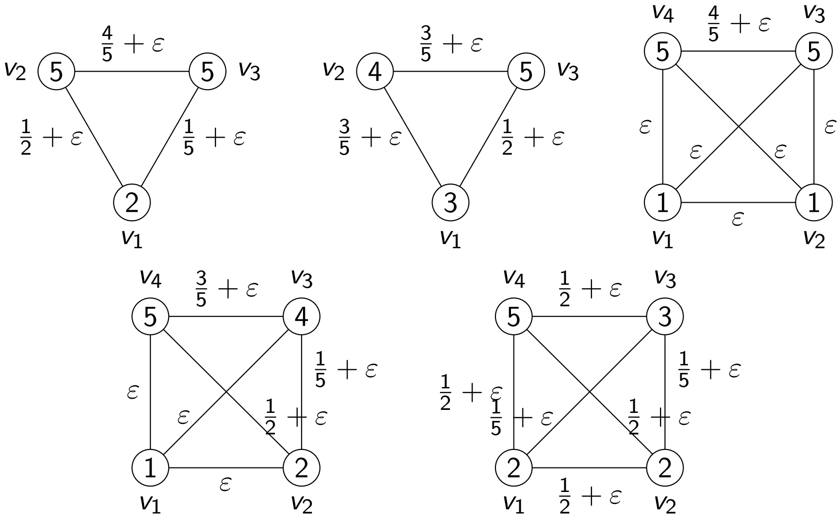
Forbidden configuration on R

εn vertices contain K_5, find K_{12} if R contains weighted triangle $v_1 v_2 v_3$.
Embedding lemma by Erdős, Hajnal, Simonovits, Sós, Szemeredi, see also Liu et. al.





All forbidden configurations for $\varrho_5(12) \leq \frac{10}{19}$



Flag algebras

Seminal paper:
Razborov, Flag Algebras, *Journal of Symbolic Logic* **72** (2007), 1239–1282.
David P. Robbins Prize by AMS for Razborov in 2013 over 300 citations (on google)

Example

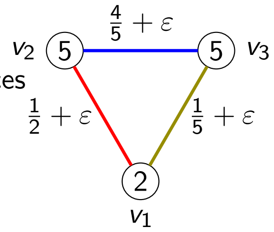
If density of edges is p, what is the minimum density of triangles?

- Designed to attack extremal problems.
- Works well if constraints as well as desired value can be computed by checking small subgraphs (or average over small subgraphs).
- The results are for the limit as graphs get very large.

Weighted problem using flag algebras

- No such thing as weighted flags
- Flag algebras allow coloring edges from a finite set of colors
- Make density ranges as colors

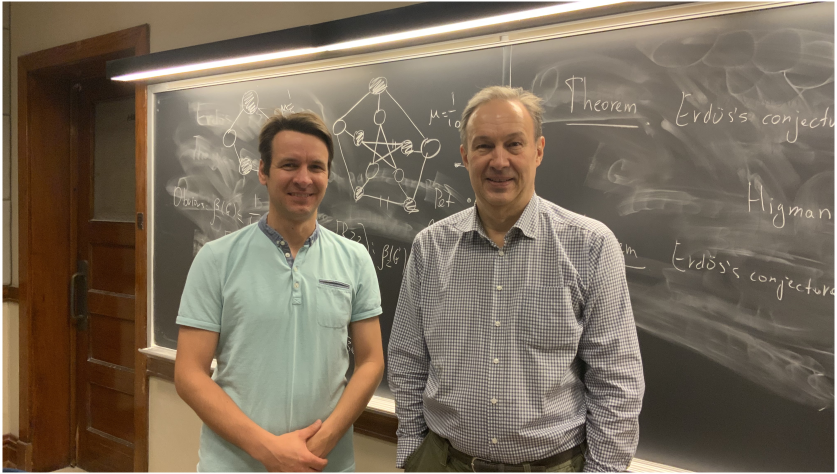
name/color	density interval	rule
1	$[0, \varepsilon)$	no embedding
2	$[\varepsilon, 1/5 + \varepsilon)$	any 1 vertex
3	$[1/5 + \varepsilon, 1/2 + \varepsilon)$	some 2 vertices
4	$[1/2 + \varepsilon, 3/5 + \varepsilon)$	any 2 vertices or some 3 vertices
5	$[3/5 + \varepsilon, 4/5 + \varepsilon)$	some 4 vertices
6	$[4/5 + \varepsilon, 1]$	any 5 vertices



Flag result

$$\frac{1}{5}c_2 + \frac{1}{2}c_3 + \frac{3}{5}c_4 + \frac{4}{5}c_5 + c_6 \leq \frac{10}{19} + o(1)$$

Flag Algebras



⋆ Nothing in these slides is endorsed by Razborov except this picture

Example extremal problem

Theorem (Mantel 1907)

Every n-vertex triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

Problem

Maximize a graph parameter (# of edges) over a class of graphs (triangle-free).

- local condition and global parameter
- threshold
- bound and extremal example

Example extremal problem

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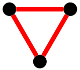
Maximize a graph parameter (# of edges) over a class of graphs (triangle-free).

- local condition and global parameter
- threshold
- bound and extremal example

We will use colors for edges and non-edges.

Flag algebras definitions

Let G be a 2-edge-colored complete graph on n vertices.



The probability that three random vertices in G span a red triangle, i.e. $\#$$/\binom{n}{3}$.

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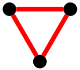

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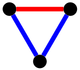

The probability that three random vertices in G span a graph isomorphic to a triangle with one red and two blue edges.

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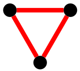

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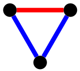

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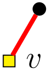

The probability that a random vertex other than v is connected to v by a red edge, i.e., the red degree of v divided by $n-1$.

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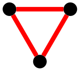

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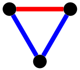
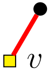


The probability that three random vertices in G span a graph isomorphic to a triangle with one red and two blue edges.



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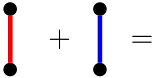

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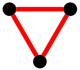

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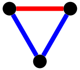

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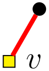

The probability that a random vertex other than v is connected to v by a red edge, i.e., the red degree of v divided by $n-1$.

 $+$  $= 1$

Flag algebras definitions

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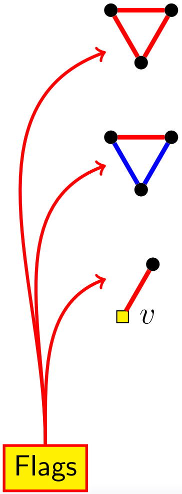

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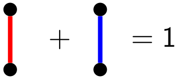
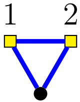
The probability that three random vertices in G span a graph isomorphic to a triangle with one red and two blue edges.

The probability that a random vertex other than v is connected to v by a red edge, i.e., the red degree of v divided by $n-1$.

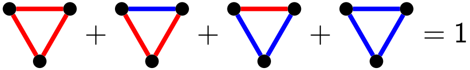




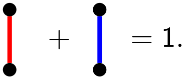
Type - flag induced by labeled vertices

Flag algebras identities

Let G be a 2-edge-colored complete graph on n vertices.



Same kind as



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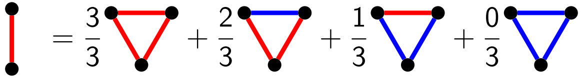
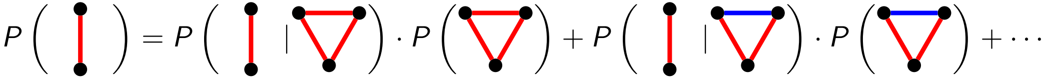


Expanded version:



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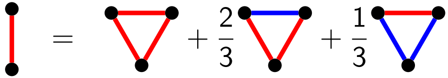

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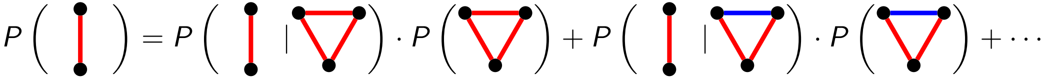

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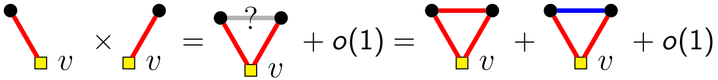

$o(1)$ as $|V(G)| \to \infty$ (will be omitted on next slides)

Flag algebras identities

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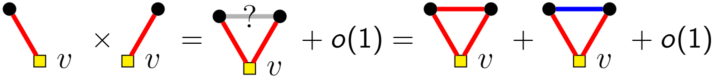

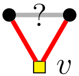

: The probability of choosing two different vertices . . .

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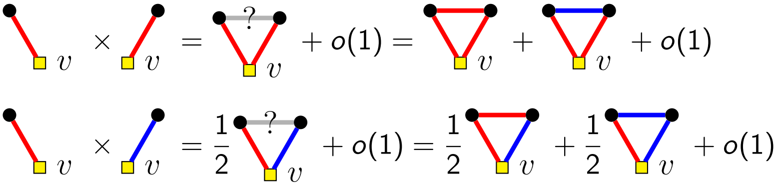

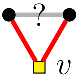

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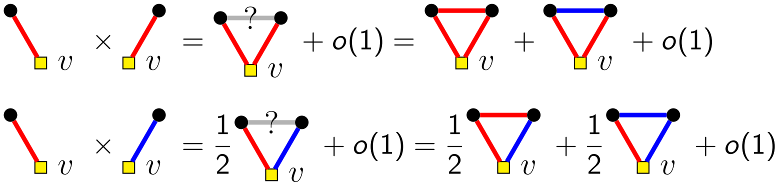

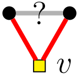

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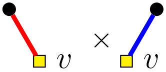
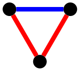
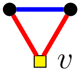


: The probability that choosing two vertices u_1, u_2 other than v gives red vu_1 and blue vu_2.

$o(1)$ as $|V(G)| \to \infty$ (will be omitted on next slides)

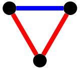
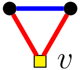
Flag algebras identities

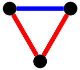
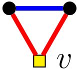
Let G be a 2-edge-colored complete graph on n vertices.

$\frac{1}{3}$  $= \frac{1}{|V(G)|} \sum_{v \in V(G)}$  v

Flag algebras identities

Let G be a 2-edge-colored complete graph on n vertices.

$\frac{1}{3}$  $= \frac{1}{|V(G)|} \sum_{v \in V(G)}$  v

 $\binom{n}{3} = \sum_{v \in V(G)}$  $v \binom{n-1}{2}$

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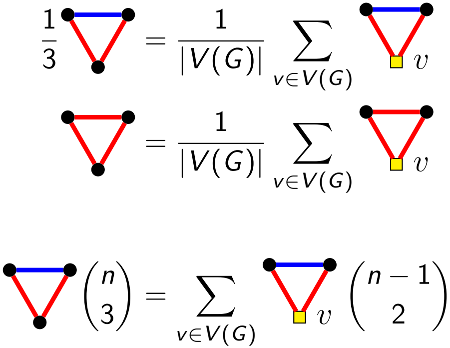

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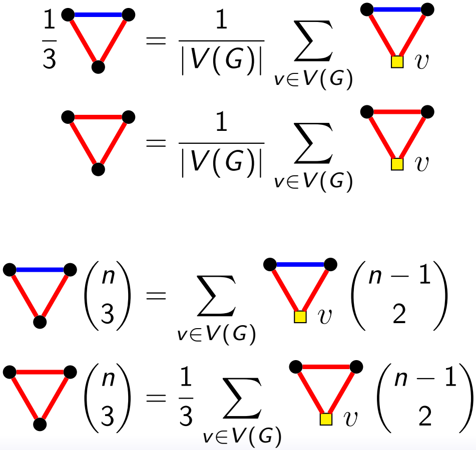
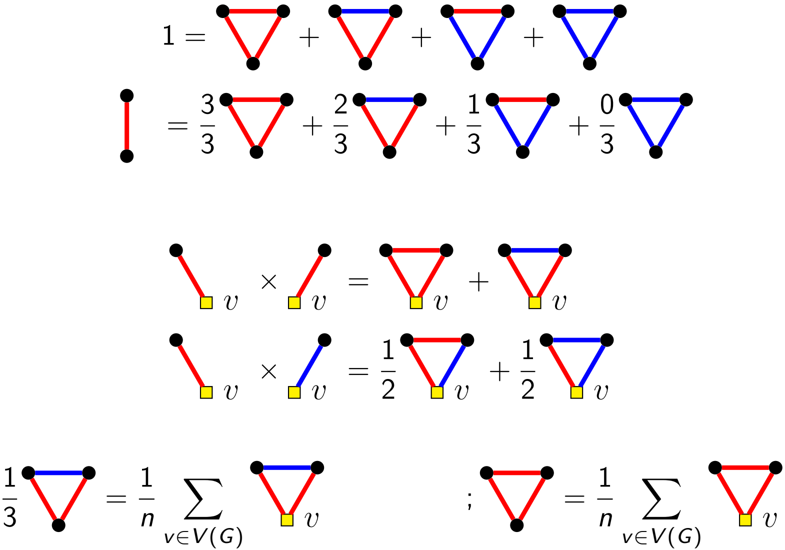


Identities Summary



Flag algebras - example

Theorem (Mantel 1907)

Every triangle-free graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2}\binom{n}{2}$ *edges.*

Assume edges are red and non-edges are blue.

Flag algebras - example

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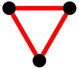
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

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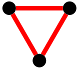
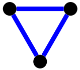
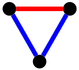
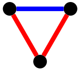
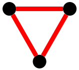
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$1 =$  $+$  $+$  $+$ 

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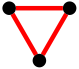
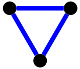
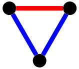
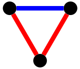
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$1 =$  $+$  $+$ 

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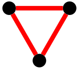
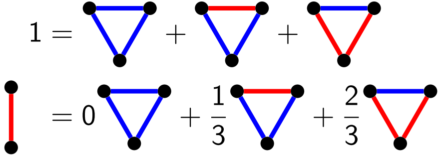
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)



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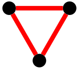
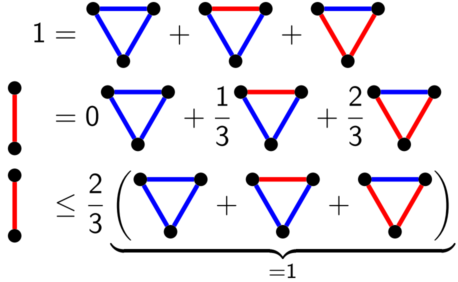
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)



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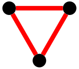
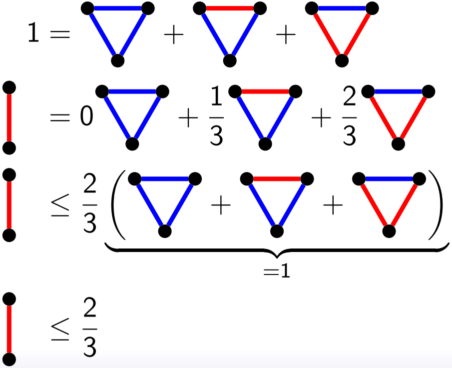
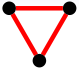
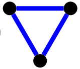
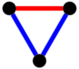
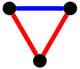
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)



Example - Mantel's theorem

Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

 $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

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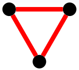
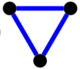
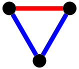
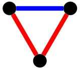
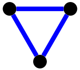
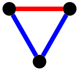
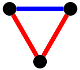
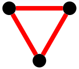
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

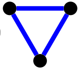
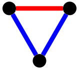
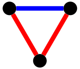
 $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

Idea: find $c_1, c_2, c_3 \in \mathbb{R}$ such that for every graph G

$0 \leq c_1$  $+ c_2$  $+ c_3$  $+ o(1)$.

Example - Mantel's theorem

Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

 $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

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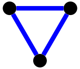
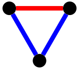
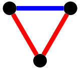
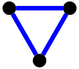
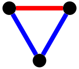
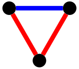
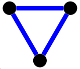
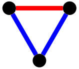
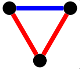
$0 \leq c_1$  $+ c_2$  $+ c_3$  $+ o(1)$.

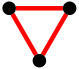
After summing together

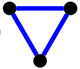
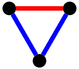
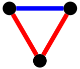
 $\leq c_1$  $+ \left(\frac{1}{3} + c_2\right)$  $+ \left(\frac{2}{3} + c_3\right)$ 

and

 $\leq \max\left\{0 + c_1, \frac{1}{3} + c_2, \frac{2}{3} + c_3\right\} \underbrace{\Big(}$  $+$  $+$  $\underbrace{\Big)}_{=1}$

Example - Mantel's theorem

Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

 $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

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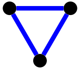
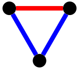
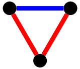
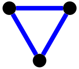
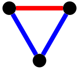
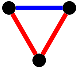
$0 \leq c_1$  $+ c_2$  $+ c_3$  $+o(1)$.

$c_3 < 0$

After summing together

 $\leq c_1$  $+ \left(\frac{1}{3} + c_2\right)$  $+ \left(\frac{2}{3} + c_3\right)$ 

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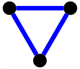
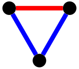
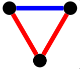
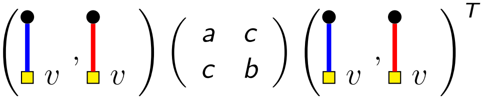
 $\leq \max\left\{0 + c_1, \frac{1}{3} + c_2, \frac{2}{3} + c_3\right\}$ $\underbrace{\left(\right.}$  $+$  $+$  $\underbrace{\left.\right)}_{=1}$

Flag algebras - candidates for c_1, c_2, c_3

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

Flag algebras - candidates for c_1, c_2, c_3

$0 \leq$



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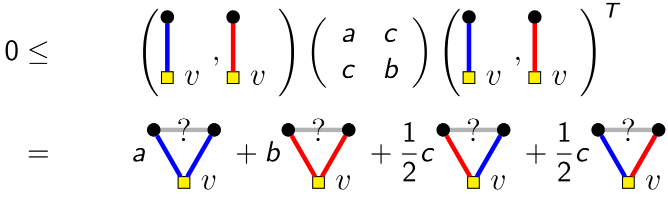
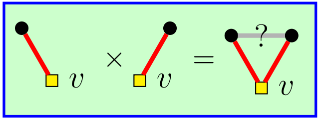
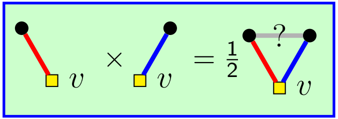
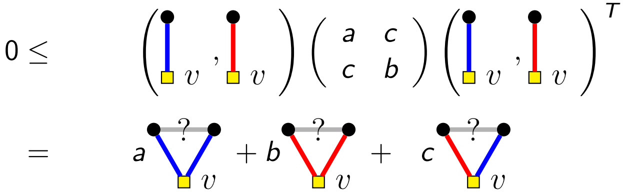
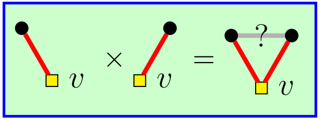


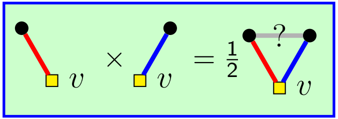




$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$ (matrix is positive semidefinite)







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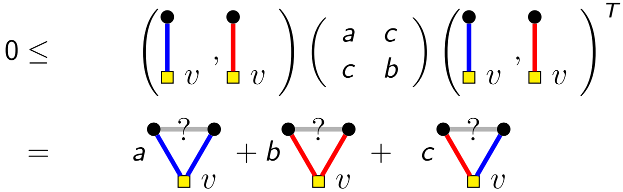

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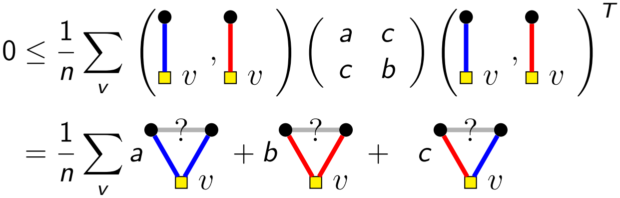

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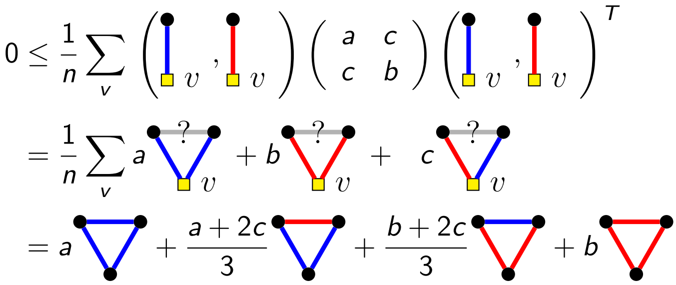
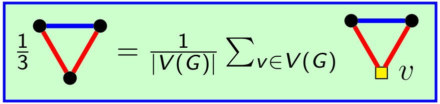
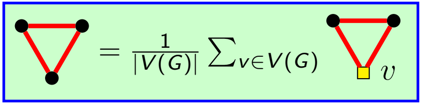
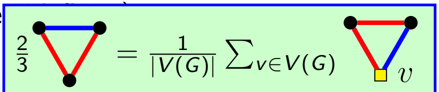






$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$ (matrix is positive se



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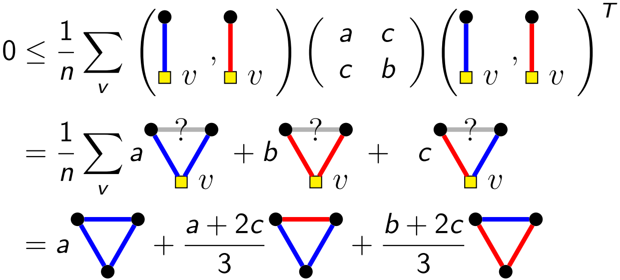

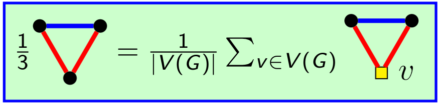

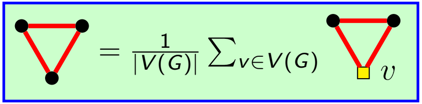

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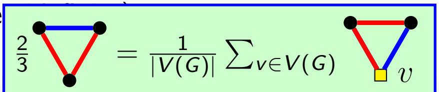

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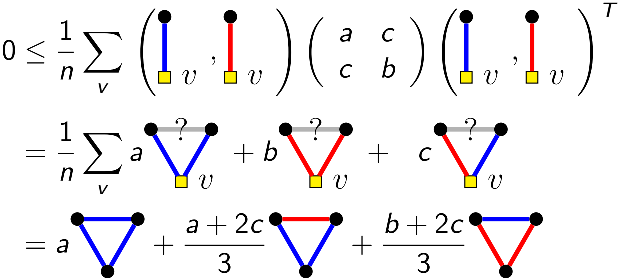
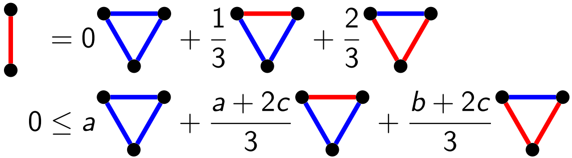


$$c_1 = a,\; c_2 = \frac{a+2c}{3},\; c_3 = \frac{b+2c}{3}$$

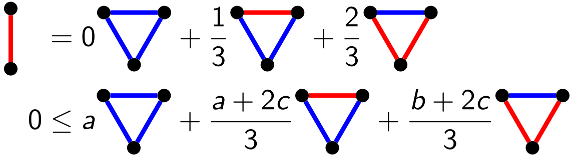
$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$ (matrix is positive semidefinite)

FLAG ALGEBRAS - USING c_1, c_2, c_3



$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

Flag algebras - using c_1, c_2, c_3



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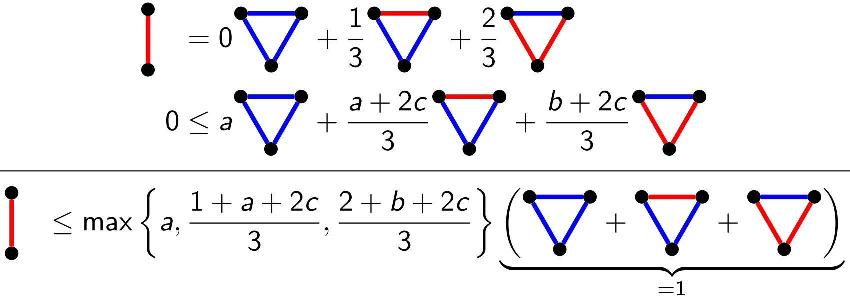
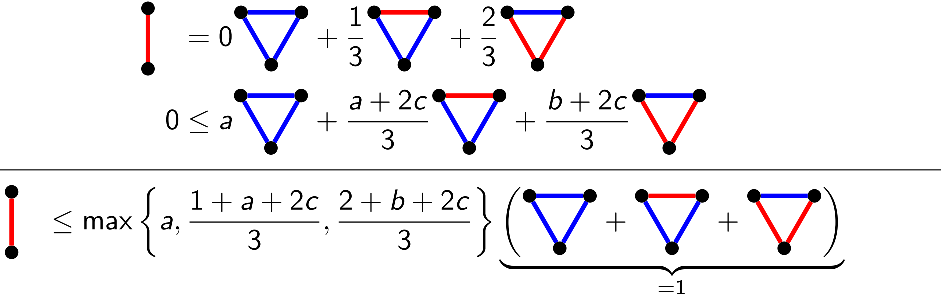
Flag algebras - using c_1, c_2, c_3

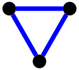
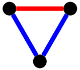
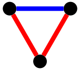


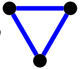
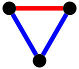
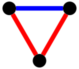
$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$$ (matrix is positive semidefinite)

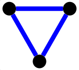
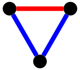
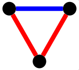


Try

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

 $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

$0 \leq a$  $+ \frac{a+2c}{3}$  $+ \frac{b+2c}{3}$ 

 $\leq \max\left\{a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3}\right\} \underbrace{\Big($  $+$  $+$  $\Big)}_{=1}$

Try

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

It gives

 $\leq \max\left\{\frac{1}{2}, \frac{1}{6}, \frac{1}{2}\right\} = \frac{1}{2}.$

Flag algebras - optimizing a, b, c

 $\leq \max\left\{a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3}\right\}$

$$(SDP)\begin{cases}\text{Minimize} & d\\ \text{subject to} & a \leq d\\ & \frac{1+a+2c}{3} \leq d\\ & \frac{2+b+2c}{3} \leq d\\ & \begin{pmatrix} a & c\\ c & b\end{pmatrix} \succcurlyeq 0\end{cases}$$

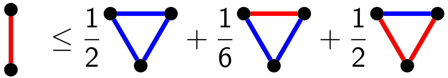
(SDP) can be solved on computers using CSDP or SDPA.
Rounding may be needed for exact results.

How to find extremal constructions?

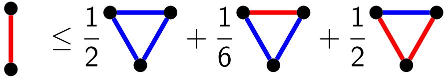
We got

$$\le \max\left\{\frac{1}{2}, \frac{1}{6}, \frac{1}{2}\right\} = \frac{1}{2}.$$

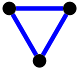
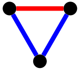
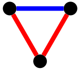
which is



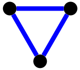
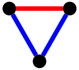
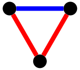
How to find extremal constructions?

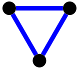
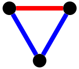
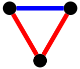


How to find extremal constructions?

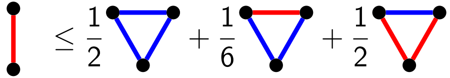
 $\leq \frac{1}{2}$  $+ \frac{1}{6}$  $+ \frac{1}{2}$ 

Suppose G is an extremal graph $\Big($  $= \frac{1}{2}\Big)$. Then

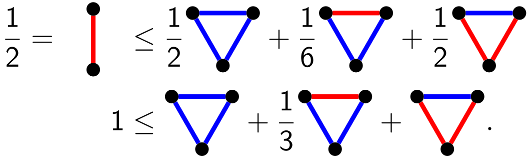
$\frac{1}{2} =$  $\leq \frac{1}{2}$  $+ \frac{1}{6}$  $+ \frac{1}{2}$ 

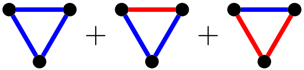
$1 \leq$  $+ \frac{1}{3}$  $+$  .

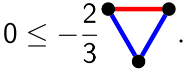
How to find extremal constructions?



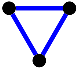
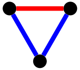
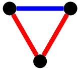
Suppose G is an extremal graph  $\left(\; = \frac{1}{2} \right)$. Then



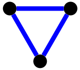
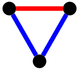
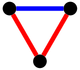
By subtracting $1 =$  we obtain

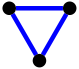
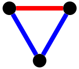
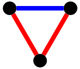


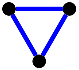
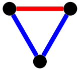
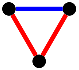
How to find extremal constructions?

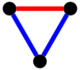
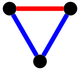
 $\leq \frac{1}{2}$  $+ \frac{1}{6}$  $+ \frac{1}{2}$ 

Suppose G is an extremal graph $\Big($  $= \frac{1}{2}\Big)$. Then

$\frac{1}{2} =$  $\leq \frac{1}{2}$  $+ \frac{1}{6}$  $+ \frac{1}{2}$ 

$1 \leq$  $+ \frac{1}{3}$  $+$  .

By subtracting $1 =$  $+$  $+$  we obtain

$0 \leq -\frac{2}{3}$  . Hence  $= 0$.

 $\leq \max\left\{\frac{1}{2}, \frac{1}{6}, \frac{1}{2}\right\} = \frac{1}{2}$

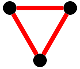
Tells us that that if ( $= \frac{1}{2}$), then

- graphs with coefficients $< \frac{1}{2}$ do not appear in any extremal example
- all subgraphs of extremal example(s) should have $\frac{1}{2}$
- gives possible subgraphs for extremal examples (if not known)
- having $\frac{1}{2}$ does not mean it appears in any extremal example

The semidefinite matrix gives a certificate.

Small experiment with an extra constraint

Mantel

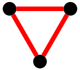
- Maximize 
- subject to  $= 0$

Solution is $\frac{1}{2}$.

Small experiment with an extra constraint

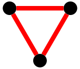
Mantel $\begin{cases} \text{Maximize} & \text{} \\ \text{subject to} & \text{} \end{cases}$

Maximize 

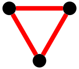
subject to  $= 0$

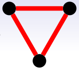
Solution is $\frac{1}{2}$. What if  $= p > \frac{1}{2}$?

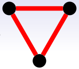
Small experiment with an extra constraint

Mantel $\begin{cases} \text{Maximize} \\ \text{subject to} \end{cases}$   $= 0$

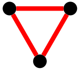
Solution is $\frac{1}{2}$. What if  $= p > \frac{1}{2}$?

$\begin{cases} \text{Minimize} \\ \text{subject to} \end{cases}$   $\geq p$

Minimize  subject to  $\geq p$.

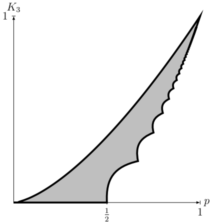
Minimize  subject to  $\geq p$.

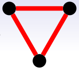
Theorem (Razborov '08)



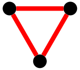
$$\geq \frac{(t-1)\left(t-2\sqrt{t(t-p(t+1))}\right)\left(t+\sqrt{t(t-p(t+1))}\right)^2}{t^2(t+1)^2}$$

where $t = \lfloor 1/(1-p) \rfloor$. *Tight bound.*



Minimize  subject to  $\geq p$.

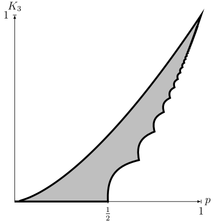
THEOREM (RAZBOROV '08)

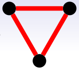
 $\geq \dfrac{(t-1)\left(t-2\sqrt{t(t-p(t+1))}\right)\left(t+\sqrt{t(t-p(t+1))}\right)^2}{t^2(t+1)^2}$

where $t = \lfloor 1/(1-p) \rfloor$. *Tight bound.*

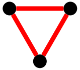
Nontrivial application of FA.

We will try a simple approach for $p = 0.6$

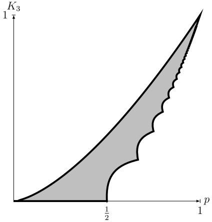


Minimize  subject to  $\geq p$.

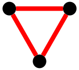
Theorem (Razborov '08)

 $$\geq \frac{(t-1)\left(t-2\sqrt{t(t-p(t+1))}\right)\left(t+\sqrt{t(t-p(t+1))}\right)^2}{t^2(t+1)^2}$$

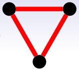
where $t = \lfloor 1/(1-p) \rfloor$. *Tight bound.*

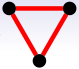


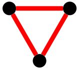
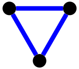
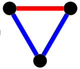
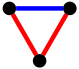
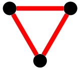
Nontrivial application of FA.
We will try a simple approach for $p = 0.6$
(We not will reproduce the result)

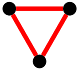
 $\geq 0.14150099\ldots$ for $p = 0.6$ by Razborov

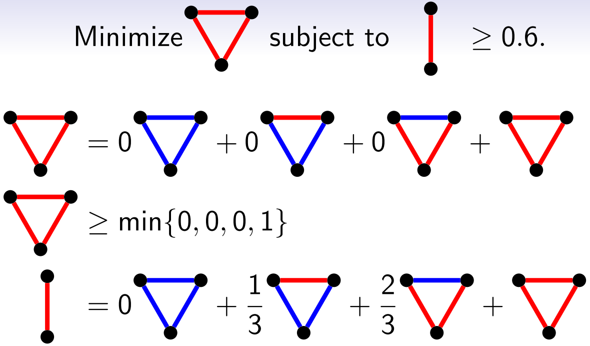
Note: Liu, Pikhurko, Staden: more exact results 2020 (99 or 144 pages)

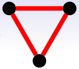
Minimize  subject to  ≥ 0.6.

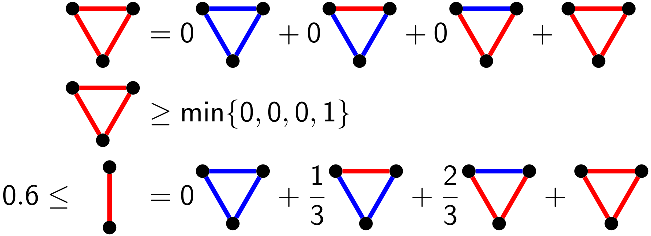
Minimize  subject to  ≥ 0.6.

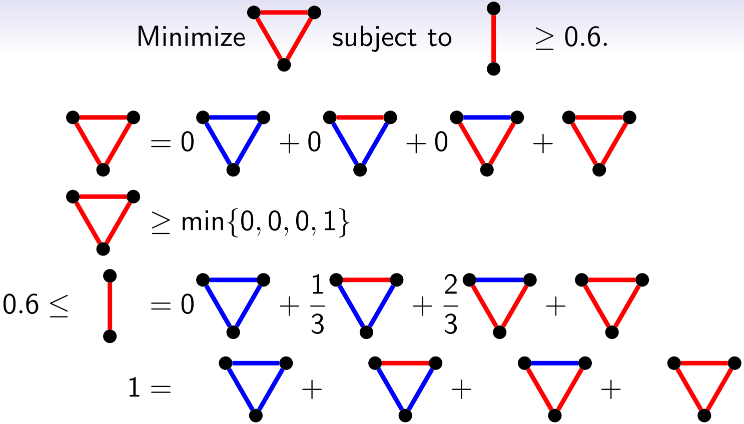
 $= 0$  $+ 0$  $+ 0$  $+$ 

 $\geq \min\{0, 0, 0, 1\}$

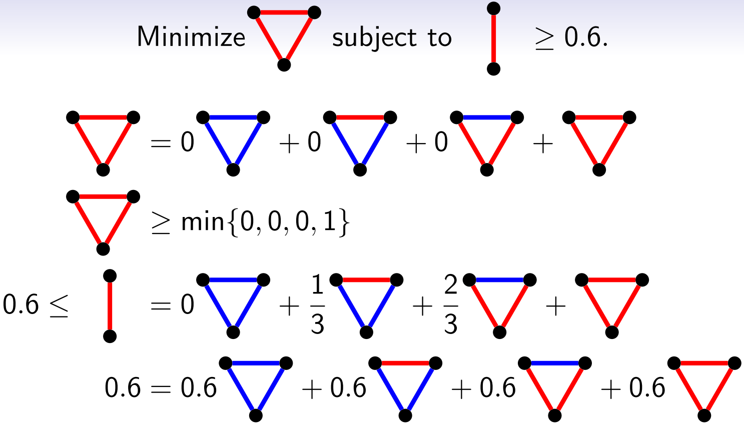

Minimize subject to ≥ 0.6.
= 0 + 0 + 0 +
≥ min{0, 0, 0, 1}
= 0 + 1/3 + 2/3 +

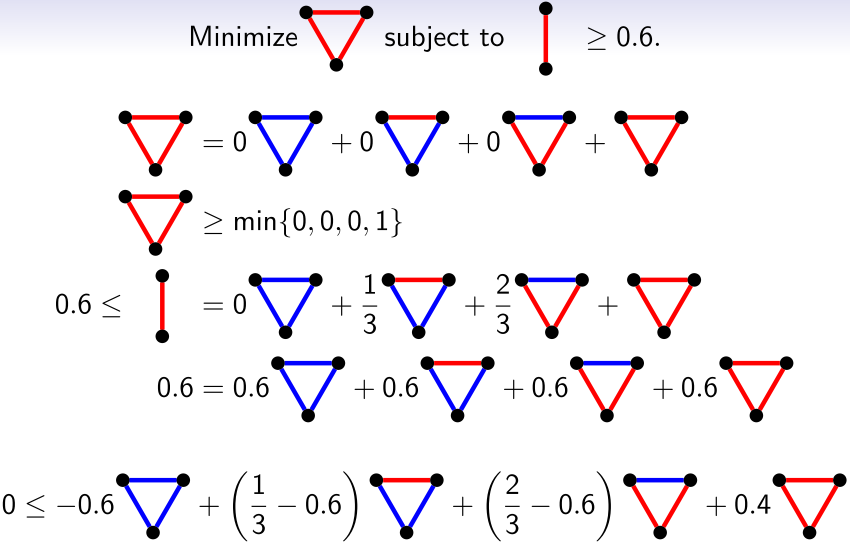
Minimize  subject to  ≥ 0.6.

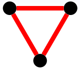
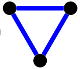
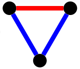
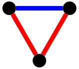
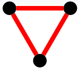


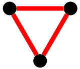


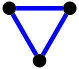
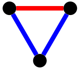
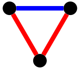
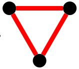
Minimize
subject to
≥ 0.6.
= 0
+ 0
+ 0
+
≥ min{0, 0, 0, 1}
0.6 ≤
= 0
+ 1/3
+ 2/3
+
1 =
+
+
+

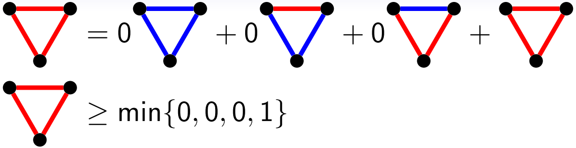

Minimize
subject to
≥ 0.6.
= 0
+ 0
+ 0
+
≥ min{0, 0, 0, 1}
0.6 ≤
= 0
+ 1/3
+ 2/3
+
0.6 = 0.6
+ 0.6
+ 0.6
+ 0.6

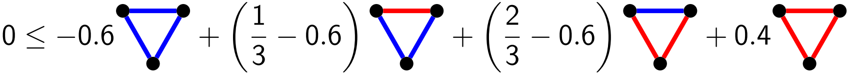

Minimize subject to ≥ 0.6.
= 0 + 0 + 0 +
≥ min{0, 0, 0, 1}
0.6 ≤ = 0 + 1/3 + 2/3 +
0.6 = 0.6 + 0.6 + 0.6 + 0.6
0 ≤ −0.6 + (1/3 − 0.6) + (2/3 − 0.6) + 0.4

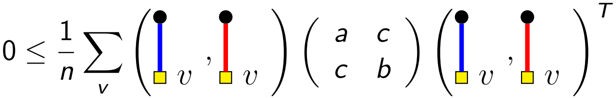
 $= 0$  $+ 0$  $+ 0$  $+$ 

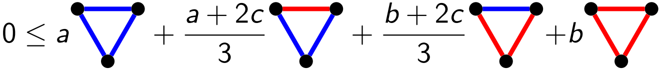
 $\geq \min\{0, 0, 0, 1\}$

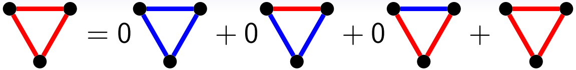
$0 \leq -0.6$  $+ \left(\frac{1}{3} - 0.6\right)$  $+ \left(\frac{2}{3} - 0.6\right)$  $+ 0.4$ 

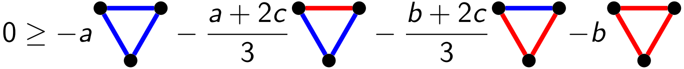

= 0 + 0 + 0 +
≥ min{0, 0, 0, 1}

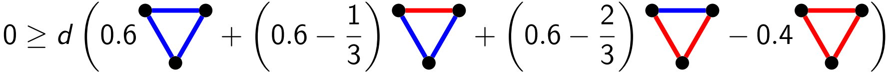

0 ≤ −0.6 + (1/3 − 0.6) + (2/3 − 0.6) + 0.4


0 ≤ 1/n Σ_v
v , v
a c
c b
v , v
T

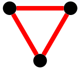
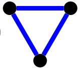
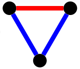
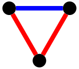
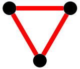

0 ≤ a + (a + 2c)/3 + (b + 2c)/3 +b

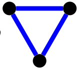
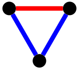
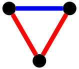
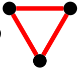


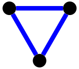
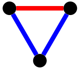
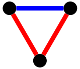
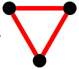


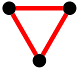


$$\begin{pmatrix} a & c & 0 \\ c & b & 0 \\ 0 & 0 & d \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

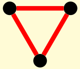
 $= 0$  $+ 0$  $+ 0$  $+$ 

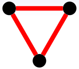
$0 \geq -a$  $- \dfrac{a+2c}{3}$  $- \dfrac{b+2c}{3}$  $-b$ 

$0 \geq d \Big(0.6$  $+ \left(0.6 - \dfrac{1}{3}\right)$  $+ \left(0.6 - \dfrac{2}{3}\right)$  $- 0.4$  $\Big)$

 $\geq \min \Big\{ 0.6d - a, \left(0.6 - \dfrac{1}{3}\right) d - \dfrac{a+2c}{3},$

$\left(0.6 - \dfrac{2}{3}\right) d - \dfrac{b+2c}{3}, 1 - 0.4d - b \Big\}$

 $\geq 0.14150099\ldots$



$$\geq \min\left\{0.6d - a, \left(0.6 - \frac{1}{3}\right)d - \frac{a+2c}{3}, \left(0.6 - \frac{2}{3}\right)d - \frac{b+2c}{3}, 1 - 0.4d - b\right\}$$

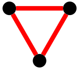
Numerical solution from CSDP:

$$a = 6 \times 0.12000065088497793850 \qquad a = 0.72$$

$$b = 6 \times 0.05333290843810910981 \qquad b = 0.32$$

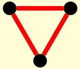
$$c = 6 \times -0.07999989818128358521 \qquad c = -0.48$$

$$d = 1.40000645402718526 5 \qquad d = 1.4$$



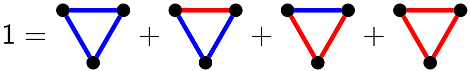
$$\geq \min\left\{0.12, 0.45\overline{3}, 0.12, 0.12\right\} = 0.12$$

How to improve 0.12?

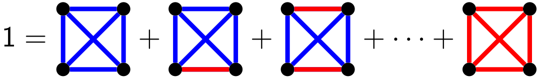


$\geq 0.14150099\ldots$

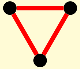
Sample bigger graphs. Instead of



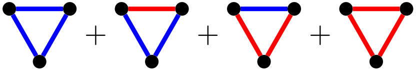
use



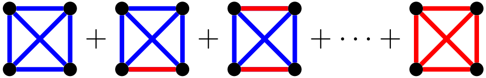
How to improve 0.12?


$\geq 0.14150099\ldots$

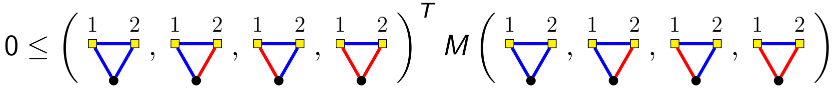
Sample bigger graphs. Instead of

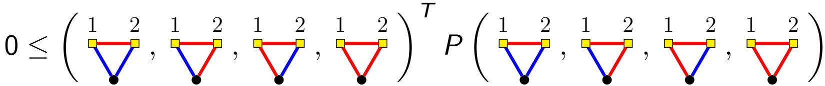
$1 =$ 

use

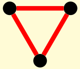
$1 =$ 

and include also $M, P \succcurlyeq 0$

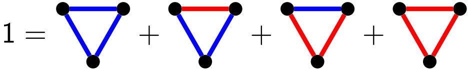




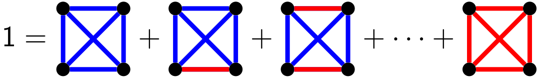
How to improve 0.12?


$\geq 0.14150099\ldots$

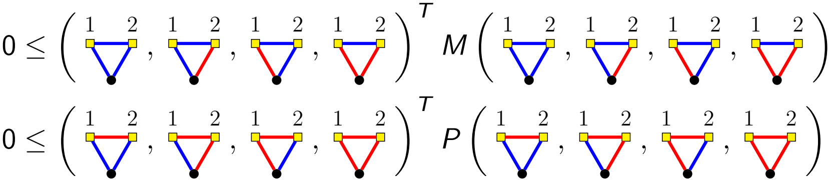
Sample bigger graphs. Instead of



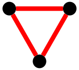
use



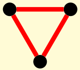
and include also $M, P \succcurlyeq 0$



This gives


$\geq 0.127815\ldots$

How to improve 0.12781...?

 $\geq 0.14150099\ldots$

Sample even bigger graphs.

Use K_5 instead of K_4

Include even more types and flags.

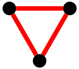
How to improve 0.12781...?

Sample even bigger graphs.

Use K_5 instead of K_4

Include even more types and flags.

This gives



$$\geq 0.1333333 = 2/15.$$

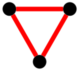
How to improve 0.12781...?

Sample even bigger graphs.

Use K_5 instead of K_4

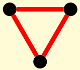
Include even more types and flags.

This gives

 $\geq 0.1333333 = 2/15.$

Try even bigger!

How to improve 0.12781. . . ?



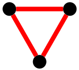
$\geq 0.14150099\ldots$

Sample even bigger graphs.

Use K_5 instead of K_4

Include even more types and flags.

This gives



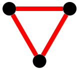
$$\geq 0.1333333 = 2/15.$$

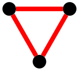
Try even bigger!

vertices	# graphs	time	bound
3	4	instant	0.12
4	11	instant	0.127815. . .
5	34	instant	0.13333. . .
6	156	seconds	0.13333. . .
7	1044	minutes	0.13333. . .
8	12346	day(s)	0.13333. . .
9	274668	not computable*	?

$\star$ needs hundreds of GB of RAM, maybe easy in 10 years?

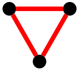
Getting 0.14150099 . . .

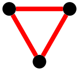
 $\geq 0.14150099\ldots$ for $p = 0.6$ by Razborov

Minimize 

subject to  $- p \geq 0$

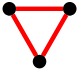
Getting 0.14150099...

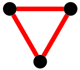
 $\geq 0.14150099\ldots$ for $p = 0.6$ by Razborov

$\begin{cases} \text{Minimize} \end{cases}$ 

subject to $H \cdot \Big($  $- p\Big) \geq 0$ for any graph H

GETTING 0.14150099 . . .

 $\geq 0.14150099\ldots$ for $p = 0.6$ by Razborov

Minimize 

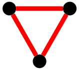
subject to $H \cdot \big($  $- p\big) \geq 0$ for any graph H

vertices	# graphs	time	bound	new bound
3	4	instant	0.12	0.12
4	11	instant	0.12781. . .	0.131746. . .
5	34	instant	0.13333. . .	0.14046241. . .
6	156	seconds	0.13333. . .	0.14150099. . .
7	1044	minutes	0.13333. . .	0.14150099. . .
8	12346	day(s)	0.13333. . .	0.14150099. . .

These are just numerical bounds! Not exact.

Goodman's bound

Recall we got for $p = 0.6$



$$\geq \min \left\{ 0.6d - a, \left(0.6 - \frac{1}{3}\right) d - \frac{a + 2c}{3}, \left(0.6 - \frac{2}{3}\right) d - \frac{b + 2c}{3}, 1 - (1 - 0.6)d - b \right\}$$

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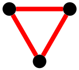
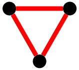


$$\geq \min\left\{0.6d - a, \left(0.6 - \frac{1}{3}\right)d - \frac{a+2c}{3}, \left(0.6 - \frac{2}{3}\right)d - \frac{b+2c}{3}, 1 - (1-0.6)d - b\right\}$$

Same thing holds when 0.6 is replaced by a parameter p.

Goodman's bound

Recall we got for $p = 0.6$



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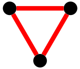
Same thing holds when 0.6 is replaced by a parameter p.

$$a = 2p^2 \qquad b = 2p^2 - 4p + 2 \qquad c = p(2p - 2) \qquad d = 4p - 1$$

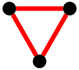
gives Goodman's bound:



$$\geq 2p^2 - p$$

Goodman's bound

Recall we got for $p = 0.6$



$$\geq \min \left\{ 0.6d - a, \left(0.6 - \frac{1}{3}\right) d - \frac{a+2c}{3}, \left(0.6 - \frac{2}{3}\right) d - \frac{b+2c}{3}, 1 - (1-0.6)d - b \right\}$$

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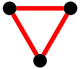

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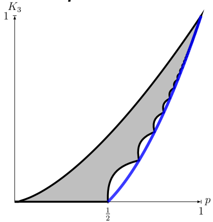

This is tight for $p \in \{\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\}$.

Thank you!