

FLAG ALGEBRAS AND WEIGHTED TURÁN PROBLEMS WITH APPLICATIONS TO RAMSEY-TURÁN QUESTIONS

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RAMSEY AND TURÁN

THEOREM (RAMSEY (1930))

For every r, s exists $R(r, s)$ such that every graph on $R(r, s)$ vertices contains K_r or \overline{K}_s .

THEOREM (TURÁN (1941))

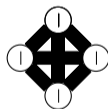
K_q -free graph on n vertices maximizing the number of edges is $T_{q-1}(n)$.



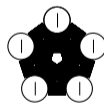
$T_2(n)$



$T_3(n)$



$T_4(n)$

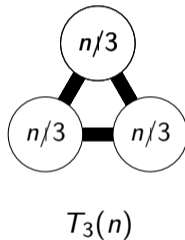


$T_5(n)$

RAMSEY-TURÁN

PROBLEM

What K_q -free graph on n vertices maximizing the number of edges while having low independence number?



PROBLEM

What K_q -free graph on n vertices is maximizing the number of edges while having low p -independence number?

p -independence number of a graph G is

$$\alpha_p(G) := \max \{ |U| : U \subseteq V(G) \text{ and } G[U] \text{ is } K_p\text{-free} \}$$

Note $\alpha_2(G) = \alpha(G)$

$Ramsey$ -Turán number

$$RT_p(n, K_q, m) := \max \{ e(G) : G \text{ is } K_q\text{-free, } v(G) = n, \alpha_p(G) \leq m, \}$$

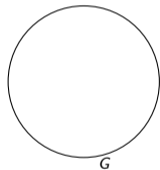
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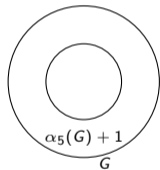
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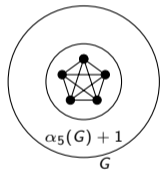
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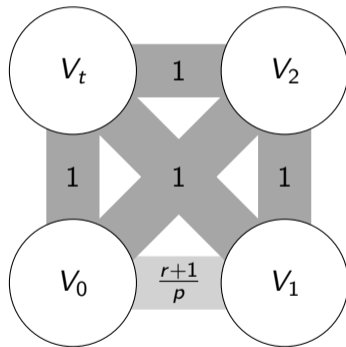
Asymptotic version

$$\varrho_p(q) := \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{RT_p(n, K_q, \varepsilon n)}{\binom{n}{2}}$$

CONJECTURE (ERDŐS, HAJNAL, SIMONOVITS, SÓS AND SZEMERÉDI '94)

The asymptotic extremal graph G for $\varrho_p(q)$ has the following structure. Let $q = pt + r + 2$, where $t \in \mathbb{N}$ and $r \in \mathbb{Z}_p$. Then there is a partition $V(G) = V_0 \cup V_1 \cup \dots \cup V_t$ such that

- $e(G[V_i]) = o(n^2)$ for all $0 \leq i \leq t$;
- $d_G(V_0, V_1) = \frac{r+1}{p} - o(1)$, and degrees in $G[V_0, V_1]$ differ by $o(n)$;
- $d_G(V_i, V_j) = 1 - o(1)$ for all pairs $\{i, j\} \neq \{0, 1\}$.



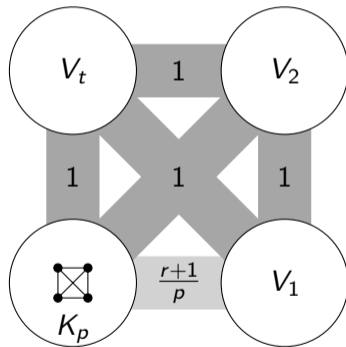
In particular

$$\varrho_p(q) = \varrho_p^*(q) := \frac{(t-1)(2p-r-1)+r+1}{t(2p-r-1)+r+1}.$$

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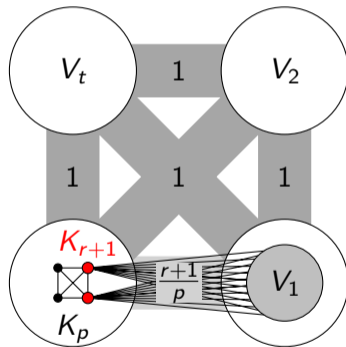
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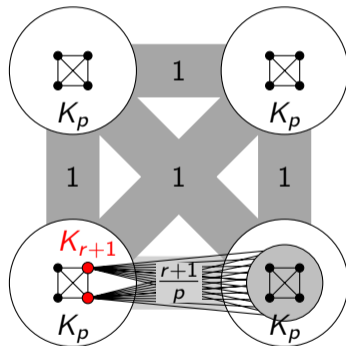
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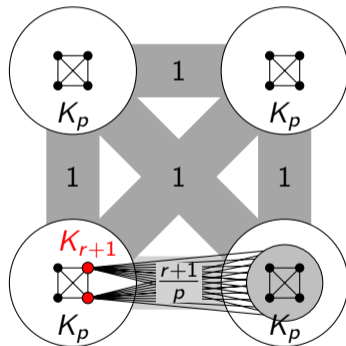
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Liu, Reiher, Sharifzadeh, and Staden $\varrho_{16}(22) = 1/6 > 5/32 = \varrho_{16}^*(22)$

CONJECTURED CONSTRUCTION

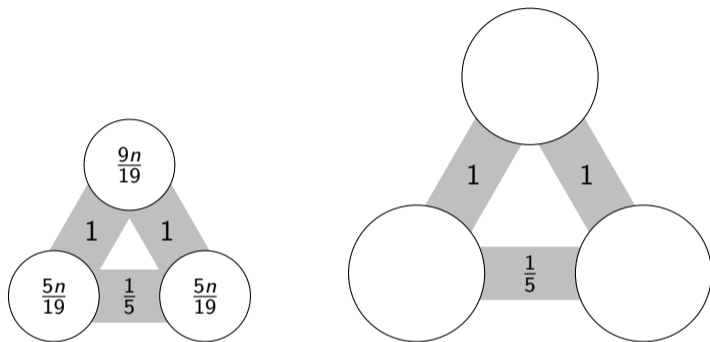


FIGURE: Sketch of a construction for $q_5(12) \geq \frac{10}{19}$.

Liu, Reiher, Sharifzadeh, and Staden

Let $q = pt + \ell + 1$. Then for all $0 \leq \ell \leq p/2$: $q_p(q) \geq q_p^*(q)$

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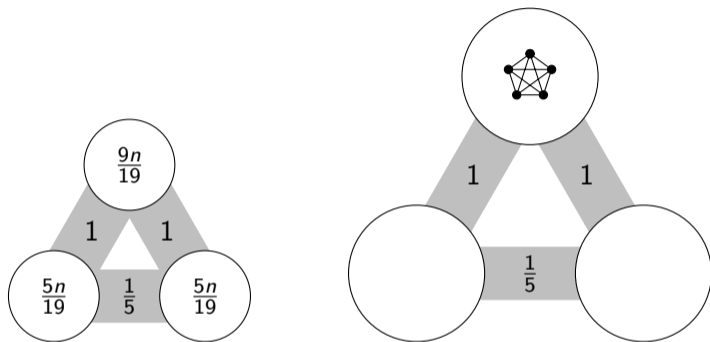


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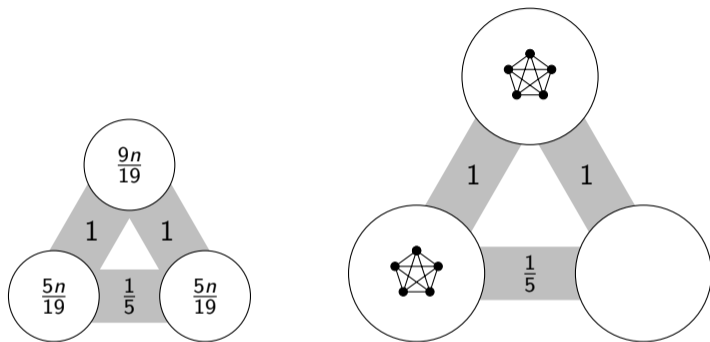


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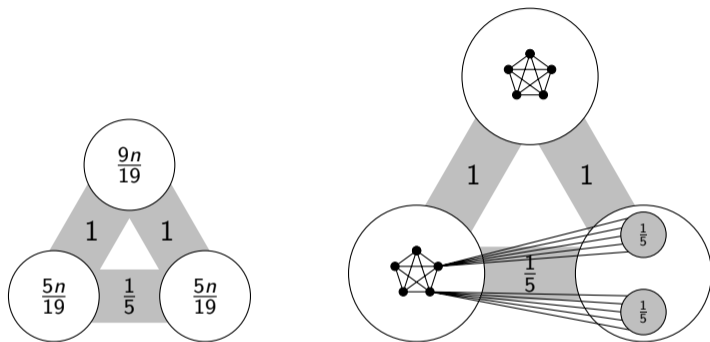


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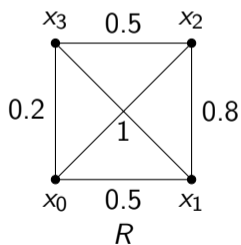
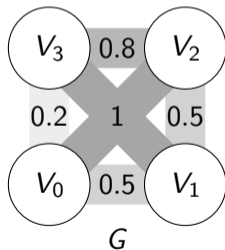
OUR WORK

We calculate upper bound on $\varrho_p(q)$ for some small values of p and q .

$$\varrho_p(q) = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{RT_p(n, K_q, \varepsilon n)}{\binom{n}{2}}.$$

Plan

- Take large K_q -free n -vertex graph G where every εn vertices contain K_p .
- Apply Szemerédi Regularity Lemma
- Get reduced graph R
- Note R is edge-weighted graph
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WEIGHTED TURÁN PROBLEMS

An *edge weighting* w is $w : E(G) \rightarrow [0, 1]$.

$$w(G) := \frac{2}{n^2} \sum_{e \in E(G)} w(e).$$

A *weighted clique* is (r, f)

$$f : \binom{[r]}{2} \rightarrow [0, 1]$$

(G, w) *contains* (r, f) if exists injective

$$\phi : [r] \rightarrow V(G) \quad \phi(i)\phi(j) \in E(G) \text{ and } w(\phi(i)\phi(j)) > f(ij)$$

ASYMPTOTIC PROBLEM

An *edge weighting* w is $w : E(G) \rightarrow [0, 1]$.

$$w(G) := \frac{2}{n^2} \sum_{e \in E(G)} w(e).$$

Asymptotic Turán problem:

$$d(K_q) := \lim_{n \rightarrow \infty} \max_{|V(G)|=n, G \text{ is } K_q\text{-free}} e(G) / \binom{n}{2}$$

Set of weighted cliques \mathcal{F}

$$d(\mathcal{F}) := \lim_{n \rightarrow \infty} \max_{|V(G)|=n, G \text{ is } \mathcal{F}\text{-free}} w(G)$$

WEIGHTED TURÁN

A *Turán edge weighting* $w_T : E(G) \rightarrow [0, 1]$.

$$w_T(e) := \frac{r}{2(r-1)} \quad \text{where } r = \operatorname{argmax}_k \{e \text{ is in } k\text{-clique in } G\}$$

$$w_T(G) := \frac{2}{n^2} \sum_{e \in E(G)} w_T(e).$$

OBSERVATION

For every $k \geq 2$

$$\lim_{n \rightarrow \infty} w_T(T_k(n)) = \frac{1}{2}$$

since $e(T_k(n)) = \frac{r-1}{r} \binom{n}{2}$.

THEOREM (BRADAČ; MALEC, TOMPKINS)

For every G holds $w_T(G) \leq \frac{1}{2}$.

KEY LEMMA

$$g(A) := \max \left\{ \mathbf{u}^T A \mathbf{u} \mid \mathbf{u} = (u_1, \dots, u_m)^T, \sum_{i=1}^m u_i = 1, u_i \geq 0 \right\}.$$

A is *dense* if for every $i \in [m]$, $A_{i,i} = 0$ and A' obtained from A by removing i^{th} row and column satisfies $g(A') < g(A)$.

LEMMA (LIU, REIHER, SHARIFZADEH, AND STADEN 2021+)

Let $m \in \mathbb{N}$ and let $A = (a_{ij})$ be a dense symmetric $m \times m$ matrix with nonnegative entries and let \mathbf{u} be optimal for A . Then

1. A is positive, that is, $a_{ij} > 0$ for every $1 \leq i < j \leq m$,
2. $u_i > 0$ for every $i \in [m]$,
3. $\sum_{i \in [m] \setminus \{j\}} a_{ij} u_i = g(A)$, for every $j \in [m]$.

THEOREM (BRADAČ)

For every G holds $w_T(G) \leq \frac{1}{2}$.

Proof: Let $V(G) = v_1, \dots, v_n$. Define $A \in \mathbb{R}^{n \times n}$

$$A_{i,j} = \begin{cases} w_T(v_i, v_j) & \text{if } (v_i, v_j) \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

for $\mathbf{x} = (1/n, \dots, 1/n)$, we obtain

$$w_T(G) = \frac{2}{n^2} \sum_{e \in E(G)} w_T(e) = \mathbf{x}^T A \mathbf{x} \leq g(A) \leq \frac{1}{2}$$

A' principal submatrix of A maximizing $g(A')$, pick minimal by inclusion
 A' is dense, let $K \subseteq V(G)$ correspond to A' .

K induces a clique by Lemma

$$a_{i,j} \leq w_T(|K|)$$

$$g(A) \leq g(A') \leq \sum_{i \in K} u_i \sum_{j \in K, j \neq i} u_j w_T(k) = w_T(k) \sum_{i \in K} u_i (1 - u_i) = w_T(k) \left(1 - \sum_{i \in K} u_i^2\right) \leq w_T(k) \left(1 - \frac{1}{k}\right) = \frac{1}{2}$$

OTHER WEIGHTS

A *clique weighting* $cw : \mathbb{N} \rightarrow [0, 1]$.

$$w(e) := cw(r) \quad \text{where } r = \operatorname{argmax}_k \{e \text{ is in } k\text{-clique in } G\}$$

$$w(G) := \frac{2}{n^2} \sum_{e \in E(G)} w(e)$$

THEOREM

Let cw be a clique weighting. Under mild assumptions, if $w(G)$ is close maximum, then G is close $T_r(n)$ for some r .

OTHER WEIGHTS

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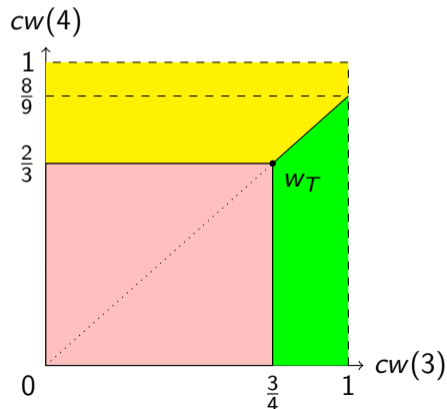
$w(G) := \frac{2}{n^2} \sum_{e \in E(G)} w(e)$

In K_5 -free graphs, $cw(2) = 1$

If $cw(3) \leq 3/4$ and $cw(4) \leq 2/3$, then $T_2(n)$ is extremal.

If $cw(3) \geq 3/4$ and $cw(3) \geq \frac{9}{8}cw(4)$, then $T_3(n)$ is extremal.

If $cw(4) \geq 2/3$ and $cw(3) \leq \frac{9}{8}cw(4)$, then $T_4(n)$ is extremal.



BACK TO RAMSEY-TURÁN

Ramsey-Turán number

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Asymptotic version

$$\varrho_p(q) = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{RT_p(n, K_q, \varepsilon n)}{\binom{n}{2}}.$$

$$\varrho_2(2t+1) = \frac{t-1}{t} \quad \text{for all } t \geq 1, \quad \text{and} \quad \varrho_2(2t) = \frac{3t-5}{3t-2} \quad \text{for all } t \geq 2.$$

p, q	5	6	7	8	9	10	11	12	13	14
3	H	S	E	H		E	H		E	H
4	0	H	S	S	E	H	*		E	H
5	0	0	S	S	S	S	E	*		
6	0	0	0	S	S	S	S	*	E	*

OUR ADDITION

$$RT_p(n, K_q, m) := \max\{e(G) : G \text{ is } K_q\text{-free, } v(G) = n, \alpha_p(G) \leq m, \}$$

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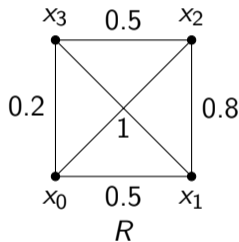
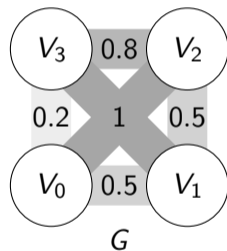
THEOREM

The following bounds hold: $\varrho_4(11) \leq \frac{4}{7}$, $\varrho_5(12) \leq \frac{10}{19}$, $\varrho_6(12) \leq \frac{5}{12}$, and $\varrho_6(14) \leq \frac{12}{23}$.
In particular, $\varrho_5(12) = \frac{10}{19}$.

Translated to weighted Turán problems solved using flag algebras.

PROOF SKETCH FOR $\rho_5(12) \leq \frac{10}{19}$.

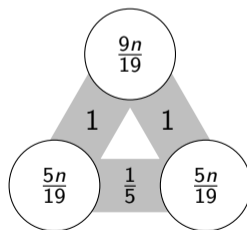
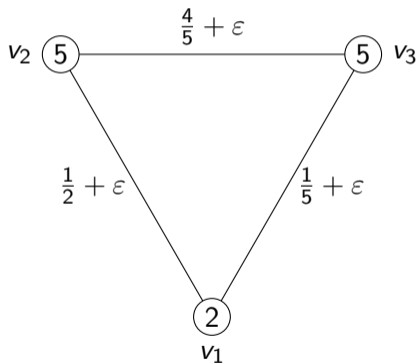
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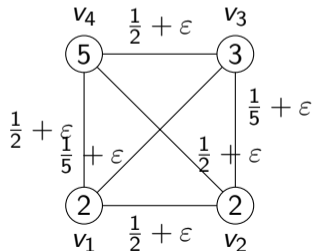
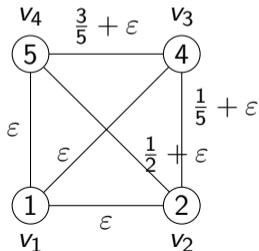
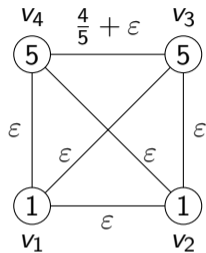
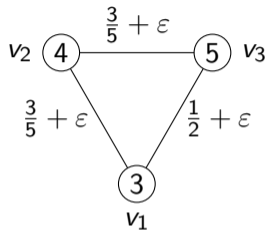
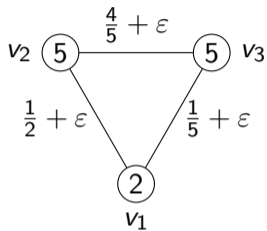
FORBIDDEN CONFIGURATION ON R

εn vertices contain K_5 , find K_{12} if R contains weighted triangle $v_1 v_2 v_3$.

Embedding lemma by Erdős, Hajnal, Simonovits, Sós, Szemerédi, see also Liu et. al.



ALL FORBIDDEN CONFIGURATIONS FOR $\rho_5(12) \leq \frac{10}{19}$



FLAG ALGEBRAS

Seminal paper:

Razborov, Flag Algebras, *Journal of Symbolic Logic* **72** (2007), 1239–1282.

David P. Robbins Prize by AMS for Razborov in 2013 over 300 citations (on google)



EXAMPLE

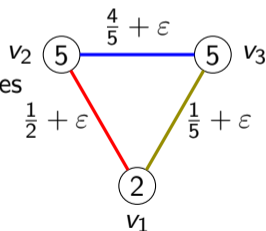
If density of edges is p , what is the minimum density of triangles?

- Designed to attack extremal problems.
- Works well if constraints as well as desired value can be computed by checking small subgraphs (or average over small subgraphs).
- The results are for the limit as graphs get very large.

WEIGHTED PROBLEM USING FLAG ALGEBRAS

- No such thing as weighted flags
- Flag algebras allow coloring edges from a finite set of colors
- Make density ranges as colors

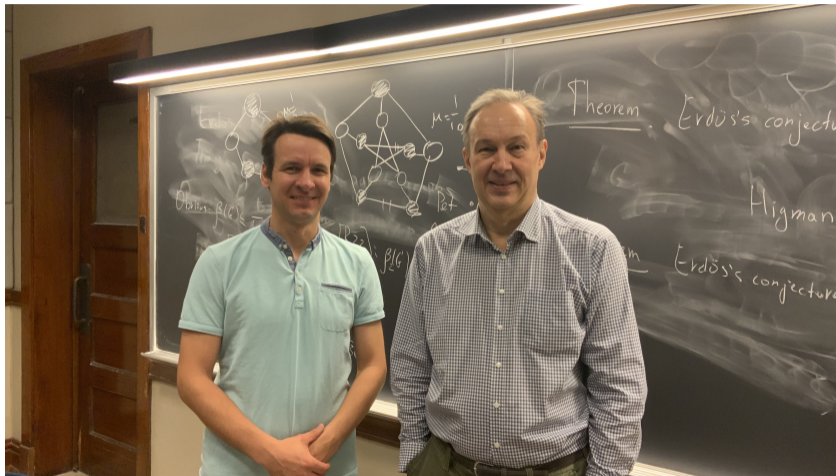
name/color	density interval	rule
1	$[0, \varepsilon)$	no embedding
2	$[\varepsilon, 1/5 + \varepsilon)$	any 1 vertex
3	$[1/5 + \varepsilon, 1/2 + \varepsilon)$	some 2 vertices
4	$[1/2 + \varepsilon, 3/5 + \varepsilon)$	any 2 vertices or some 3 vertices
5	$[3/5 + \varepsilon, 4/5 + \varepsilon)$	some 4 vertices
6	$[4/5 + \varepsilon, 1]$	any 5 vertices



Flag result

$$\frac{1}{5}c_2 + \frac{1}{2}c_3 + \frac{3}{5}c_4 + \frac{4}{5}c_5 + c_6 \leq \frac{10}{19} + o(1)$$

FLAG ALGEBRAS



★ Nothing in these slides is endorsed by Razborov except this picture

EXAMPLE EXTREMAL PROBLEM

THEOREM (MANTEL 1907)

Every n -vertex triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

PROBLEM

Maximize a graph parameter (# of edges) over a class of graphs (triangle-free).

- local condition and global parameter
- threshold
- bound and extremal example

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- local condition and global parameter
- threshold
- bound and extremal example

We will use colors for **edges** and **non-edges**.

FLAG ALGEBRAS DEFINITIONS

Let G be a 2-edge-colored complete graph on n vertices.



The probability that three random vertices in G span a red triangle, i.e. $\# \triangle / \binom{n}{3}$.

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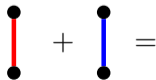
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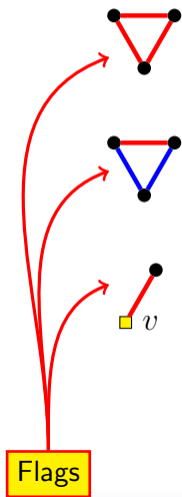


The probability that a random vertex other than v is connected to v by a red edge, i.e., the red degree of v divided by $n - 1$.

$$\text{red edge} + \text{blue edge} = 1$$

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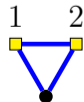


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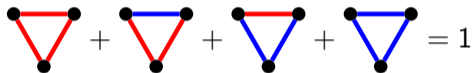
$$\text{red edge} + \text{blue edge} = 1$$



Type - flag induced by labeled vertices

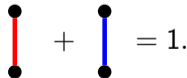
FLAG ALGEBRAS IDENTITIES

Let G be a 2-edge-colored complete graph on n vertices.



The diagram shows an equation with four terms on the left and an equals sign followed by a 1 on the right. Each term is a triangle with three vertices and three edges. The first triangle has all three edges colored red. The second triangle has the top edge colored blue and the two bottom edges colored red. The third triangle has the top edge colored red and the two bottom edges colored blue. The fourth triangle has all three edges colored blue. The triangles are arranged horizontally and separated by plus signs.

Same kind as



The diagram shows an equation with two terms on the left and an equals sign followed by a 1 on the right. Each term is a vertical edge connecting two vertices. The first edge is colored red, and the second edge is colored blue. The edges are arranged horizontally and separated by a plus sign.

FLAG ALGEBRAS IDENTITIES

Let G be a 2-edge-colored complete graph on n vertices.

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} = \frac{3}{3} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \frac{0}{3} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array}$$

Expanded version:

$$P \left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right) = P \left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \mid \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right) \cdot P \left(\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right) + P \left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \mid \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right) \cdot P \left(\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right) + \dots$$

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FLAG ALGEBRAS IDENTITIES

Let G be a 2-edge-colored complete graph on n vertices.

$$\begin{array}{c} \bullet \\ \text{red} \\ \square v \end{array} \times \begin{array}{c} \bullet \\ \text{red} \\ \square v \end{array} = \begin{array}{c} \bullet \text{ ? } \bullet \\ \text{red} \quad \text{red} \\ \square v \end{array} + o(1) = \begin{array}{c} \bullet \text{ red } \bullet \\ \text{red} \quad \text{red} \\ \square v \end{array} + \begin{array}{c} \bullet \text{ blue } \bullet \\ \text{red} \quad \text{red} \\ \square v \end{array} + o(1)$$

$o(1)$ as $|V(G)| \rightarrow \infty$ (will be omitted on next slides)

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: The probability of choosing two **different** vertices ...

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
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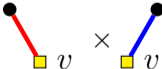
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 : The probability of choosing two **different** vertices ...


 : The probability that choosing two vertices u_1, u_2 other than v gives red vu_1 and blue vu_2 .

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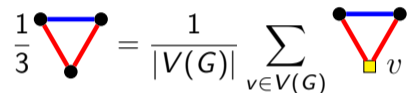
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Let G be a 2-edge-colored complete graph on n vertices.

$$\frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \square v \end{array}$$

FLAG ALGEBRAS IDENTITIES

Let G be a 2-edge-colored complete graph on n vertices.

$$\frac{1}{3} \text{triangle} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \text{triangle}_v$$


$$\text{triangle} \binom{n}{3} = \sum_{v \in V(G)} \text{triangle}_v \binom{n-1}{2}$$


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IDENTITIES SUMMARY

$$1 = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$$

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$$\frac{1}{3} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \frac{1}{n} \sum_{v \in V(G)} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \square v \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$$

$$; \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \frac{1}{n} \sum_{v \in V(G)} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \square v \end{array}$$

FLAG ALGEBRAS - EXAMPLE

THEOREM (MANTEL 1907)

Every triangle-free graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2} \binom{n}{2}$ edges.



Assume **edges are red** and **non-edges are blue**.

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$$1 = \text{triangle with 3 blue edges} + \text{triangle with 2 blue edges and 1 red edge} + \text{triangle with 1 blue edge and 2 red edges}$$

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$$1 = \begin{array}{c} \text{triangle with 3 blue edges} \\ + \\ \text{triangle with 2 blue edges and 1 red edge} \\ + \\ \text{triangle with 1 blue edge and 2 red edges} \end{array}$$
$$\begin{array}{c} \text{edge with 2 red edges} \\ = 0 \cdot \begin{array}{c} \text{triangle with 3 blue edges} \\ + \\ \frac{1}{3} \cdot \begin{array}{c} \text{triangle with 2 blue edges and 1 red edge} \\ + \\ \frac{2}{3} \cdot \begin{array}{c} \text{triangle with 1 blue edge and 2 red edges} \end{array} \end{array} \end{array}$$

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Assume edges are red and non-edges are blue.

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)



$$\begin{aligned}
 1 &= \text{triangle (all blue)} + \text{triangle (top blue, bottom red)} + \text{triangle (top red, bottom blue)} \\
 \text{edge (both red)} &= 0 \cdot \text{triangle (all blue)} + \frac{1}{3} \cdot \text{triangle (top blue, bottom red)} + \frac{2}{3} \cdot \text{triangle (top red, bottom blue)} \\
 \text{edge (both red)} &\leq \frac{2}{3} \left(\underbrace{\text{triangle (all blue)} + \text{triangle (top blue, bottom red)} + \text{triangle (top red, bottom blue)}}_{=1} \right)
 \end{aligned}$$

FLAG ALGEBRAS - EXAMPLE

THEOREM (MANTEL 1907)



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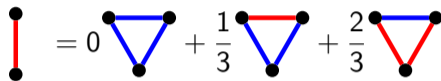
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$$\begin{aligned}
 1 &= \text{triangle (all blue)} + \text{triangle (2 blue, 1 red)} + \text{triangle (all red)} \\
 \text{edge (all red)} &= 0 \cdot \text{triangle (all blue)} + \frac{1}{3} \cdot \text{triangle (2 blue, 1 red)} + \frac{2}{3} \cdot \text{triangle (all red)} \\
 &\leq \frac{2}{3} \left(\text{triangle (all blue)} + \text{triangle (2 blue, 1 red)} + \text{triangle (all red)} \right) \\
 &\qquad\qquad\qquad \underbrace{\hspace{10em}}_{=1} \\
 \text{edge (all red)} &\leq \frac{2}{3}
 \end{aligned}$$





EXAMPLE - MANTEL'S THEOREM

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$\text{Edge} = 0 \cdot \text{Triangle} + \frac{1}{3} \cdot \text{Triangle} + \frac{2}{3} \cdot \text{Triangle}$$


EXAMPLE - MANTEL'S THEOREM



Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$\text{Edge} = 0 \cdot \text{Triangle} + \frac{1}{3} \cdot \text{Triangle} + \frac{2}{3} \cdot \text{Triangle}$$
 = 0  + $\frac{1}{3}$  + $\frac{2}{3}$ 

Idea: find $c_1, c_2, c_3 \in \mathbb{R}$ such that for every graph G

$$0 \leq c_1 \cdot \text{Triangle} + c_2 \cdot \text{Triangle} + c_3 \cdot \text{Triangle} + o(1).$$
 + c_2  + c_3 

EXAMPLE - MANTEL'S THEOREM

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$\text{red edge} = 0 \cdot \text{blue triangle} + \frac{1}{3} \cdot \text{triangle with 1 red edge} + \frac{2}{3} \cdot \text{triangle with 2 red edges}$$

Idea: find $c_1, c_2, c_3 \in \mathbb{R}$ such that for every graph G

$$0 \leq c_1 \cdot \text{blue triangle} + c_2 \cdot \text{triangle with 1 red edge} + c_3 \cdot \text{triangle with 2 red edges} + o(1).$$



After summing together

$$\text{red edge} \leq c_1 \cdot \text{blue triangle} + \left(\frac{1}{3} + c_2\right) \cdot \text{triangle with 1 red edge} + \left(\frac{2}{3} + c_3\right) \cdot \text{triangle with 2 red edges}$$

and

$$\text{red edge} \leq \max \left\{ 0 + c_1, \frac{1}{3} + c_2, \frac{2}{3} + c_3 \right\} \underbrace{\left(\text{blue triangle} + \text{triangle with 1 red edge} + \text{triangle with 2 red edges} \right)}_{=1}$$

EXAMPLE - MANTEL'S THEOREM

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$\text{red edge} = 0 \cdot \text{blue triangle} + \frac{1}{3} \cdot \text{triangle with 1 red edge} + \frac{2}{3} \cdot \text{red triangle}$$

Idea: find $c_1, c_2, c_3 \in \mathbb{R}$ such that for every graph G

$$0 \leq c_1 \cdot \text{blue triangle} + c_2 \cdot \text{triangle with 1 red edge} + c_3 \cdot \text{red triangle} + o(1).$$



After summing together

$$\text{red edge} \leq c_1 \cdot \text{blue triangle} + \left(\frac{1}{3} + c_2\right) \cdot \text{triangle with 1 red edge} + \left(\frac{2}{3} + c_3\right) \cdot \text{red triangle}$$

and

$$\text{red edge} \leq \max \left\{ 0 + c_1, \frac{1}{3} + c_2, \frac{2}{3} + c_3 \right\} \underbrace{\left(\text{blue triangle} + \text{triangle with 1 red edge} + \text{red triangle} \right)}_{=1}$$

FLAG ALGEBRAS - CANDIDATES FOR c_1, c_2, c_3

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

FLAG ALGEBRAS - CANDIDATES FOR c_1, c_2, c_3

$$0 \leq \left(\begin{array}{c|c} \bullet & \bullet \\ \hline \text{blue} & \text{red} \\ \hline \text{yellow} & \text{yellow} \end{array} v, \begin{array}{c|c} \bullet & \bullet \\ \hline \text{red} & \text{red} \\ \hline \text{yellow} & \text{yellow} \end{array} v \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c|c} \bullet & \bullet \\ \hline \text{blue} & \text{red} \\ \hline \text{yellow} & \text{yellow} \end{array} v, \begin{array}{c|c} \bullet & \bullet \\ \hline \text{red} & \text{red} \\ \hline \text{yellow} & \text{yellow} \end{array} v \right)^T$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

FLAG ALGEBRAS - CANDIDATES FOR c_1, c_2, c_3

$$\begin{aligned}
 0 &\leq \begin{pmatrix} \bullet & \bullet \\ \square v & \square v \end{pmatrix} \begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} \bullet & \bullet \\ \square v & \square v \end{pmatrix}^T \\
 &= a \begin{matrix} \bullet & ? \\ \diagdown & \diagup \\ \square & \square v \end{matrix} + b \begin{matrix} \bullet & ? \\ \diagup & \diagdown \\ \square & \square v \end{matrix} + \frac{1}{2}c \begin{matrix} \bullet & ? \\ \diagdown & \diagup \\ \square & \square v \end{matrix} + \frac{1}{2}c \begin{matrix} \bullet & ? \\ \diagup & \diagdown \\ \square & \square v \end{matrix}
 \end{aligned}$$

$$\begin{matrix} \bullet & \bullet \\ \diagdown & \diagup \\ \square & \square v \end{matrix} \times \begin{matrix} \bullet & \bullet \\ \diagup & \diagdown \\ \square & \square v \end{matrix} = \begin{matrix} \bullet & ? \\ \diagdown & \diagup \\ \square & \square v \end{matrix}$$

$$\begin{matrix} \bullet & \bullet \\ \diagdown & \diagup \\ \square & \square v \end{matrix} \times \begin{matrix} \bullet & \bullet \\ \diagup & \diagdown \\ \square & \square v \end{matrix} = \frac{1}{2} \begin{matrix} \bullet & ? \\ \diagdown & \diagup \\ \square & \square v \end{matrix}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

FLAG ALGEBRAS - CANDIDATES FOR c_1, c_2, c_3

$$\begin{aligned}
 0 &\leq \left(\begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \right)^T \\
 &= a \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ / \quad \backslash \\ \square v \end{array} + b \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ / \quad \backslash \\ \square v \end{array} + c \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ / \quad \backslash \\ \square v \end{array}
 \end{aligned}$$

$$\begin{array}{c} \bullet \\ | \\ \square v \end{array} \times \begin{array}{c} \bullet \\ | \\ \square v \end{array} = \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ / \quad \backslash \\ \square v \end{array}$$

$$\begin{array}{c} \bullet \\ | \\ \square v \end{array} \times \begin{array}{c} \bullet \\ | \\ \square v \end{array} = \frac{1}{2} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ / \quad \backslash \\ \square v \end{array}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

FLAG ALGEBRAS - CANDIDATES FOR c_1, c_2, c_3

$$\begin{aligned}
 0 &\leq \left(\begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \right)^T \\
 &= a \begin{array}{c} \bullet \quad ? \quad \bullet \\ \diagdown \quad \diagup \\ \square v \end{array} + b \begin{array}{c} \bullet \quad ? \quad \bullet \\ \diagdown \quad \diagup \\ \square v \end{array} + c \begin{array}{c} \bullet \quad ? \quad \bullet \\ \diagdown \quad \diagup \\ \square v \end{array}
 \end{aligned}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

FLAG ALGEBRAS - CANDIDATES FOR c_1, c_2, c_3

$$\begin{aligned}
 0 &\leq \frac{1}{n} \sum_v \left(\begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \right)^T \\
 &= \frac{1}{n} \sum_v a \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \square v \end{array} + b \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \square v \end{array} + c \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \square v \end{array}
 \end{aligned}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

FLAG ALGEBRAS - CANDIDATES FOR c_1, c_2, c_3

$$\begin{aligned}
 0 &\leq \frac{1}{n} \sum_v \begin{pmatrix} \bullet \\ \square v \end{pmatrix} \begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} \bullet \\ \square v \end{pmatrix}^T \\
 &= \frac{1}{n} \sum_v a \begin{matrix} \bullet & ? & \bullet \\ \diagdown & & / \\ \square v \end{matrix} + b \begin{matrix} \bullet & ? & \bullet \\ / & & \diagdown \\ \square v \end{matrix} + c \begin{matrix} \bullet & ? & \bullet \\ / & & \diagdown \\ \square v \end{matrix} \\
 &= a \begin{matrix} \bullet & \bullet & \bullet \\ \diagdown & & / \\ \square v \end{matrix} + \frac{a+2c}{3} \begin{matrix} \bullet & \bullet & \bullet \\ / & & \diagdown \\ \square v \end{matrix} + \frac{b+2c}{3} \begin{matrix} \bullet & \bullet & \bullet \\ \diagdown & & / \\ \square v \end{matrix} + b \begin{matrix} \bullet & \bullet & \bullet \\ / & & \diagdown \\ \square v \end{matrix}
 \end{aligned}$$

$$\frac{1}{3} \begin{matrix} \bullet & \bullet & \bullet \\ / & & \diagdown \\ \square v \end{matrix} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{matrix} \bullet & \bullet & \bullet \\ / & & \diagdown \\ \square v \end{matrix}$$

$$\begin{matrix} \bullet & \bullet & \bullet \\ / & & \diagdown \\ \square v \end{matrix} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{matrix} \bullet & \bullet & \bullet \\ / & & \diagdown \\ \square v \end{matrix}$$

$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$ (matrix is positive semi-definite)

$$\frac{2}{3} \begin{matrix} \bullet & \bullet & \bullet \\ / & & \diagdown \\ \square v \end{matrix} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{matrix} \bullet & \bullet & \bullet \\ / & & \diagdown \\ \square v \end{matrix}$$

FLAG ALGEBRAS - CANDIDATES FOR c_1, c_2, c_3

$$\begin{aligned}
 0 &\leq \frac{1}{n} \sum_v \left(\begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \right)^T \\
 &= \frac{1}{n} \sum_v a \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ | \\ \square v \end{array} + b \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ | \\ \square v \end{array} + c \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ | \\ \square v \end{array} \\
 &= a \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ | \\ \bullet \end{array} + \frac{a+2c}{3} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ | \\ \bullet \end{array} + \frac{b+2c}{3} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ | \\ \bullet \end{array}
 \end{aligned}$$

$$\frac{1}{3} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ | \\ \bullet \end{array} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ | \\ \square v \end{array}$$

$$\begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ | \\ \bullet \end{array} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ | \\ \square v \end{array}$$

$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$ (matrix is positive semi-definite)

$$\frac{2}{3} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ | \\ \bullet \end{array} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ | \\ \square v \end{array}$$

FLAG ALGEBRAS - CANDIDATES FOR c_1, c_2, c_3

$$\begin{aligned}
 0 &\leq \frac{1}{n} \sum_v \left(\begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \right)^T \\
 &= \frac{1}{n} \sum_v a \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \square v \end{array} + b \begin{array}{c} \bullet \text{---} \bullet \\ \diagup \quad \diagdown \\ \square v \end{array} + c \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \square v \end{array} \\
 &= a \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{a+2c}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{b+2c}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \\
 c_1 &= a, \quad c_2 = \frac{a+2c}{3}, \quad c_3 = \frac{b+2c}{3}
 \end{aligned}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

FLAG ALGEBRAS - USING c_1, c_2, c_3

$$\begin{aligned} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} &= 0 \begin{array}{c} \bullet \quad \bullet \\ \backslash \quad / \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \quad \bullet \\ / \quad \backslash \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \quad \bullet \\ \backslash \quad / \\ \bullet \end{array} \\ 0 \leq a &\begin{array}{c} \bullet \quad \bullet \\ \backslash \quad / \\ \bullet \end{array} + \frac{a+2c}{3} \begin{array}{c} \bullet \quad \bullet \\ / \quad \backslash \\ \bullet \end{array} + \frac{b+2c}{3} \begin{array}{c} \bullet \quad \bullet \\ \backslash \quad / \\ \bullet \end{array} \end{aligned}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

FLAG ALGEBRAS - USING c_1, c_2, c_3

$$\begin{aligned} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} &= 0 \begin{array}{c} \bullet & \bullet \\ \backslash & / \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet & \bullet \\ / & \backslash \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet & \bullet \\ \backslash & / \\ \bullet \end{array} \\ 0 \leq a & \begin{array}{c} \bullet & \bullet \\ \backslash & / \\ \bullet \end{array} + \frac{a+2c}{3} \begin{array}{c} \bullet & \bullet \\ / & \backslash \\ \bullet \end{array} + \frac{b+2c}{3} \begin{array}{c} \bullet & \bullet \\ \backslash & / \\ \bullet \end{array} \end{aligned}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

FLAG ALGEBRAS - USING c_1, c_2, c_3

$$\begin{aligned}
 \text{Diagram} &= 0 \cdot \text{Diagram}_1 + \frac{1}{3} \cdot \text{Diagram}_2 + \frac{2}{3} \cdot \text{Diagram}_3 \\
 0 \leq a &+ \frac{a+2c}{3} \cdot \text{Diagram}_1 + \frac{b+2c}{3} \cdot \text{Diagram}_3
 \end{aligned}$$

(Note: The diagrams are triangles with vertices at the top and bottom. The top edge is blue in all. The bottom-left edge is blue in Diagram 1 and Diagram 3, and red in Diagram 2. The bottom-right edge is red in Diagram 2 and Diagram 3, and blue in Diagram 1.)

$$\text{Diagram} \leq \max \left\{ a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3} \right\} \underbrace{\left(\text{Diagram}_1 + \text{Diagram}_2 + \text{Diagram}_3 \right)}_{=1}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

FLAG ALGEBRAS - USING c_1, c_2, c_3

$$\begin{aligned}
 \text{Diagram} &= 0 \cdot \text{Diagram}_1 + \frac{1}{3} \cdot \text{Diagram}_2 + \frac{2}{3} \cdot \text{Diagram}_3 \\
 0 \leq a &+ \frac{a+2c}{3} \cdot \text{Diagram}_2 + \frac{b+2c}{3} \cdot \text{Diagram}_3
 \end{aligned}$$

$$\text{Diagram} \leq \max \left\{ a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3} \right\} \underbrace{\left(\text{Diagram}_1 + \text{Diagram}_2 + \text{Diagram}_3 \right)}_{=1}$$

Try

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

FLAG ALGEBRAS - USING c_1, c_2, c_3

$$\begin{aligned}
 \text{Diagram} &= 0 \cdot \text{Diagram}_1 + \frac{1}{3} \cdot \text{Diagram}_2 + \frac{2}{3} \cdot \text{Diagram}_3 \\
 0 \leq a &+ \frac{a+2c}{3} \cdot \text{Diagram}_2 + \frac{b+2c}{3} \cdot \text{Diagram}_3
 \end{aligned}$$

$$\text{Diagram} \leq \max \left\{ a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3} \right\} \underbrace{\left(\text{Diagram}_1 + \text{Diagram}_2 + \text{Diagram}_3 \right)}_{=1}$$

Try

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

It gives

$$\text{Diagram} \leq \max \left\{ \frac{1}{2}, \frac{1}{6}, \frac{1}{2} \right\} = \frac{1}{2}.$$

FLAG ALGEBRAS - OPTIMIZING a, b, c

$$\bullet \leq \max \left\{ a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3} \right\}$$

$$(SDP) \left\{ \begin{array}{l} \text{Minimize } d \\ \text{subject to } a \leq d \\ \frac{1+a+2c}{3} \leq d \\ \frac{2+b+2c}{3} \leq d \\ \begin{pmatrix} a & c \\ c & b \end{pmatrix} \succeq 0 \end{array} \right.$$

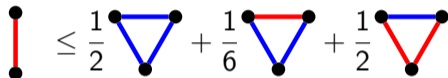
(SDP) can be solved on computers using CSDP or SDPA.
Rounding may be needed for exact results.

HOW TO FIND EXTREMAL CONSTRUCTIONS?

We got

$$\text{---} \leq \max \left\{ \frac{1}{2}, \frac{1}{6}, \frac{1}{2} \right\} = \frac{1}{2}.$$

which is

$$\text{---} \leq \frac{1}{2} \text{---} + \frac{1}{6} \text{---} + \frac{1}{2} \text{---}$$


HOW TO FIND EXTREMAL CONSTRUCTIONS?

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \leq \frac{1}{2} \begin{array}{c} \bullet & \bullet \\ \backslash & / \\ & \bullet \end{array} + \frac{1}{6} \begin{array}{c} \bullet & \bullet \\ \backslash & / \\ & \bullet \end{array} + \frac{1}{2} \begin{array}{c} \bullet & \bullet \\ \backslash & / \\ & \bullet \end{array}$$

The diagram illustrates an inequality between graph structures. On the left is a single vertical red edge connecting two black vertices. This is shown to be less than or equal to the sum of three terms, each representing a triangle with two black vertices at the top and one at the bottom. The first term is a triangle with a blue top edge and blue bottom edges, multiplied by $\frac{1}{2}$. The second term is a triangle with a red top edge and blue bottom edges, multiplied by $\frac{1}{6}$. The third term is a triangle with a blue top edge and red bottom edges, multiplied by $\frac{1}{2}$.

HOW TO FIND EXTREMAL CONSTRUCTIONS?

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \leq \frac{1}{2} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \frac{1}{6} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \frac{1}{2} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$$

Suppose G is an extremal graph $\left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} = \frac{1}{2} \right)$. Then

$$\begin{array}{c} \frac{1}{2} = \\ \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \end{array} \leq \frac{1}{2} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \frac{1}{6} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \frac{1}{2} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$$

$$1 \leq \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} .$$

HOW TO FIND EXTREMAL CONSTRUCTIONS?

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \leq \frac{1}{2} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \frac{1}{6} \begin{array}{c} \bullet \\ \diagup \quad \text{red} \quad \diagdown \\ \bullet \end{array} + \frac{1}{2} \begin{array}{c} \bullet \\ \text{red} \quad \diagdown \quad \diagup \\ \bullet \end{array}$$

Suppose G is an extremal graph $\left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} = \frac{1}{2} \right)$. Then

$$\begin{aligned} \frac{1}{2} = \begin{array}{c} \bullet \\ | \\ \bullet \end{array} &\leq \frac{1}{2} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \frac{1}{6} \begin{array}{c} \bullet \\ \diagup \quad \text{red} \quad \diagdown \\ \bullet \end{array} + \frac{1}{2} \begin{array}{c} \bullet \\ \text{red} \quad \diagdown \quad \diagup \\ \bullet \end{array} \\ 1 &\leq \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \\ \diagup \quad \text{red} \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \text{red} \quad \diagdown \quad \diagup \\ \bullet \end{array}. \end{aligned}$$

By subtracting $1 = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagup \quad \text{red} \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \text{red} \quad \diagdown \quad \diagup \\ \bullet \end{array}$ we obtain

$$0 \leq -\frac{2}{3} \begin{array}{c} \bullet \\ \diagup \quad \text{red} \quad \diagdown \\ \bullet \end{array}.$$

HOW TO FIND EXTREMAL CONSTRUCTIONS?

$$\text{red edge} \leq \frac{1}{2} \text{blue triangle} + \frac{1}{6} \text{mixed triangle} + \frac{1}{2} \text{red triangle}$$

Suppose G is an extremal graph $\left(\text{red edge} = \frac{1}{2} \right)$. Then

$$\begin{aligned} \frac{1}{2} &= \text{red edge} \leq \frac{1}{2} \text{blue triangle} + \frac{1}{6} \text{mixed triangle} + \frac{1}{2} \text{red triangle} \\ 1 &\leq \text{blue triangle} + \frac{1}{3} \text{mixed triangle} + \text{red triangle} \end{aligned}$$

By subtracting $1 = \text{blue triangle} + \text{mixed triangle} + \text{red triangle}$ we obtain

$$0 \leq -\frac{2}{3} \text{mixed triangle} \text{ . Hence } \text{mixed triangle} = 0.$$


$$\text{---} \leq \max \left\{ \frac{1}{2}, \frac{1}{6}, \frac{1}{2} \right\} = \frac{1}{2}$$

Tells us that that if $\left(\text{---} = \frac{1}{2} \right)$, then

- graphs with coefficients $< \frac{1}{2}$ do not appear in any extremal example
- all subgraphs of extremal example(s) should have $\frac{1}{2}$
- gives possible subgraphs for extremal examples (if not known)
- having $\frac{1}{2}$ does not mean it appears in any extremal example

The semidefinite matrix gives a certificate.

SMALL EXPERIMENT WITH AN EXTRA CONSTRAINT

$$\text{Mantel} \left\{ \begin{array}{l} \text{Maximize} \\ \text{subject to} \end{array} \right. \left. \begin{array}{c} \text{Diagram} \\ = 0 \end{array} \right.$$


Solution is $\frac{1}{2}$.

SMALL EXPERIMENT WITH AN EXTRA CONSTRAINT

$$\text{Mantel} \begin{cases} \text{Maximize} \\ \text{subject to} \end{cases} \begin{array}{c} \bullet \\ | \\ \bullet \\ \triangle \\ \bullet \end{array} = 0$$



Solution is $\frac{1}{2}$. What if $\begin{array}{c} \bullet \\ | \\ \bullet \end{array} = p > \frac{1}{2}$?



SMALL EXPERIMENT WITH AN EXTRA CONSTRAINT

$$\text{Mantel} \begin{cases} \text{Maximize} \\ \text{subject to} \end{cases} \begin{array}{c} \bullet \\ | \\ \bullet \\ \triangle \\ \bullet \end{array} = 0$$

Solution is $\frac{1}{2}$. What if $\begin{array}{c} \bullet \\ | \\ \bullet \end{array} = p > \frac{1}{2}$?

$$\begin{cases} \text{Minimize} \\ \text{subject to} \end{cases} \begin{array}{c} \triangle \\ \bullet \\ | \\ \bullet \end{array} \geq p$$

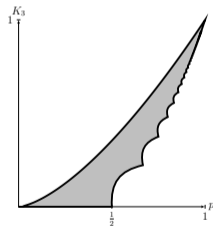
Minimize  subject to  $\geq p$.



Minimize  subject to  $\geq p$.

THEOREM (RAZBOROV '08)

$$\text{triangle} \geq \frac{(t-1) \left(t - 2\sqrt{t(t-p(t+1))} \right) \left(t + \sqrt{t(t-p(t+1))} \right)^2}{t^2(t+1)^2}$$

where $t = \lfloor 1/(1-p) \rfloor$. Tight bound.



Minimize  subject to  $\geq p$.

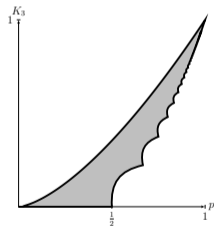
THEOREM (RAZBOROV '08)



$$\text{triangle} \geq \frac{(t-1) \left(t - 2\sqrt{t(t-p(t+1))} \right) \left(t + \sqrt{t(t-p(t+1))} \right)^2}{t^2(t+1)^2}$$

where $t = \lfloor 1/(1-p) \rfloor$. *Tight bound.*

Nontrivial application of FA.

We will try a simple approach for $p = 0.6$



Minimize  subject to  $\geq p$.

THEOREM (RAZBOROV '08)

$$\text{triangle} \geq \frac{(t-1) \left(t - 2\sqrt{t(t-p(t+1))} \right) \left(t + \sqrt{t(t-p(t+1))} \right)^2}{t^2(t+1)^2}$$

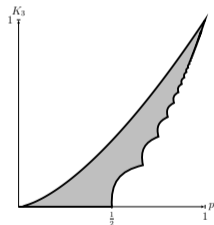
where $t = \lfloor 1/(1-p) \rfloor$. *Tight bound.*

Nontrivial application of FA.



We will try a simple approach for $p = 0.6$


(We not will reproduce the result)






$$\text{triangle} \geq 0.14150099 \dots \text{ for } p = 0.6 \text{ by Razborov}$$





Note: Liu, Pikhurko, Staden: more exact results 2020 (99 or 144 pages)

Minimize  subject to  ≥ 0.6 .

Minimize  subject to  ≥ 0.6 .

 = 0  + 0  + 0  + 


 $\geq \min\{0, 0, 0, 1\}$

Minimize  subject to  ≥ 0.6 .

$$\begin{array}{c} \text{triangle with red edges} \end{array} = 0 \begin{array}{c} \text{triangle with blue edges} \end{array} + 0 \begin{array}{c} \text{triangle with blue edges} \end{array} + 0 \begin{array}{c} \text{triangle with red edges} \end{array} + \begin{array}{c} \text{triangle with red edges} \end{array}$$

$$\begin{array}{c} \text{triangle with red edges} \end{array} \geq \min\{0, 0, 0, 1\}$$

$$\begin{array}{c} \text{edge with red edge} \end{array} = 0 \begin{array}{c} \text{triangle with blue edges} \end{array} + \frac{1}{3} \begin{array}{c} \text{triangle with blue edges} \end{array} + \frac{2}{3} \begin{array}{c} \text{triangle with red edges} \end{array} + \begin{array}{c} \text{triangle with red edges} \end{array}$$

Minimize  subject to  ≥ 0.6 .

$$\begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} = 0 \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} + 0 \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} + 0 \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array}$$

$$\begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \geq \min\{0, 0, 0, 1\}$$

$$0.6 \leq \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} = 0 \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array}$$


Minimize  subject to  ≥ 0.6 .

$$\text{triangle with red edges} = 0 \cdot \text{triangle with blue edges} + 0 \cdot \text{triangle with blue edges} + 0 \cdot \text{triangle with red edges} + \text{triangle with red edges}$$

$$\text{triangle with red edges} \geq \min\{0, 0, 0, 1\}$$

$$0.6 \leq \text{edge with red edge} = 0 \cdot \text{triangle with blue edges} + \frac{1}{3} \cdot \text{triangle with blue edges} + \frac{2}{3} \cdot \text{triangle with red edges} + \text{triangle with red edges}$$

$$1 = \text{triangle with blue edges} + \text{triangle with blue edges} + \text{triangle with red edges} + \text{triangle with red edges}$$



Minimize  subject to  ≥ 0.6 .

$$\begin{matrix} \bullet & & \bullet \\ \diagdown & & / \\ \bullet & & \bullet \end{matrix} = 0 \begin{matrix} \bullet & & \bullet \\ \diagdown & & / \\ \bullet & & \bullet \end{matrix} + 0 \begin{matrix} \bullet & & \bullet \\ \diagdown & & / \\ \bullet & & \bullet \end{matrix} + 0 \begin{matrix} \bullet & & \bullet \\ \diagdown & & / \\ \bullet & & \bullet \end{matrix} + \begin{matrix} \bullet & & \bullet \\ \diagdown & & / \\ \bullet & & \bullet \end{matrix}$$

$$\begin{matrix} \bullet & & \bullet \\ \diagdown & & / \\ \bullet & & \bullet \end{matrix} \geq \min\{0, 0, 0, 1\}$$

$$0.6 \leq \begin{matrix} \bullet & & \bullet \\ \diagdown & & / \\ \bullet & & \bullet \end{matrix} = 0 \begin{matrix} \bullet & & \bullet \\ \diagdown & & / \\ \bullet & & \bullet \end{matrix} + \frac{1}{3} \begin{matrix} \bullet & & \bullet \\ \diagdown & & / \\ \bullet & & \bullet \end{matrix} + \frac{2}{3} \begin{matrix} \bullet & & \bullet \\ \diagdown & & / \\ \bullet & & \bullet \end{matrix} + \begin{matrix} \bullet & & \bullet \\ \diagdown & & / \\ \bullet & & \bullet \end{matrix}$$

$$0.6 = 0.6 \begin{matrix} \bullet & & \bullet \\ \diagdown & & / \\ \bullet & & \bullet \end{matrix} + 0.6 \begin{matrix} \bullet & & \bullet \\ \diagdown & & / \\ \bullet & & \bullet \end{matrix} + 0.6 \begin{matrix} \bullet & & \bullet \\ \diagdown & & / \\ \bullet & & \bullet \end{matrix} + 0.6 \begin{matrix} \bullet & & \bullet \\ \diagdown & & / \\ \bullet & & \bullet \end{matrix}$$

Minimize  subject to  ≥ 0.6 .

$$\text{triangle (red)} = 0 \cdot \text{triangle (blue)} + 0 \cdot \text{triangle (blue)} + 0 \cdot \text{triangle (red)} + \text{triangle (red)}$$

$$\text{triangle (red)} \geq \min\{0, 0, 0, 1\}$$

$$0.6 \leq \text{edge} = 0 \cdot \text{triangle (blue)} + \frac{1}{3} \cdot \text{triangle (blue)} + \frac{2}{3} \cdot \text{triangle (red)} + \text{triangle (red)}$$

$$0.6 = 0.6 \cdot \text{triangle (blue)} + 0.6 \cdot \text{triangle (blue)} + 0.6 \cdot \text{triangle (red)} + 0.6 \cdot \text{triangle (red)}$$

$$0 \leq -0.6 \cdot \text{triangle (blue)} + \left(\frac{1}{3} - 0.6\right) \cdot \text{triangle (blue)} + \left(\frac{2}{3} - 0.6\right) \cdot \text{triangle (red)} + 0.4 \cdot \text{triangle (red)}$$

$$\begin{array}{c}
 \text{Red Triangle} \\
 \text{Red Triangle}
 \end{array}
 = 0 \begin{array}{c} \text{Blue Triangle} \\ \text{Blue Triangle} \end{array}
 + 0 \begin{array}{c} \text{Blue Triangle} \\ \text{Blue Triangle} \end{array}
 + 0 \begin{array}{c} \text{Red Triangle} \\ \text{Red Triangle} \end{array}
 + \begin{array}{c} \text{Red Triangle} \\ \text{Red Triangle} \end{array}$$

$$\begin{array}{c} \text{Red Triangle} \\ \text{Red Triangle} \end{array}
 \geq \min\{0, 0, 0, 1\}$$

$$0 \leq -0.6 \begin{array}{c} \text{Blue Triangle} \\ \text{Blue Triangle} \end{array}
 + \left(\frac{1}{3} - 0.6\right) \begin{array}{c} \text{Blue Triangle} \\ \text{Blue Triangle} \end{array}
 + \left(\frac{2}{3} - 0.6\right) \begin{array}{c} \text{Red Triangle} \\ \text{Red Triangle} \end{array}
 + 0.4 \begin{array}{c} \text{Red Triangle} \\ \text{Red Triangle} \end{array}$$

$$\begin{array}{c}
 \text{Red Triangle} \\
 \text{Blue Triangle} \\
 \text{Red Triangle} \\
 \text{Red Triangle}
 \end{array}
 = 0 \text{ Blue Triangle} + 0 \text{ Blue Triangle} + 0 \text{ Red Triangle} + \text{Red Triangle}$$

$$\begin{array}{c}
 \text{Red Triangle} \\
 \text{Red Triangle}
 \end{array}
 \geq \min\{0, 0, 0, 1\}$$

$$0 \leq -0.6 \begin{array}{c} \text{Blue Triangle} \\ \text{Blue Triangle} \end{array} + \left(\frac{1}{3} - 0.6\right) \begin{array}{c} \text{Red Triangle} \\ \text{Blue Triangle} \end{array} + \left(\frac{2}{3} - 0.6\right) \begin{array}{c} \text{Blue Triangle} \\ \text{Red Triangle} \end{array} + 0.4 \begin{array}{c} \text{Red Triangle} \\ \text{Red Triangle} \end{array}$$

$$0 \leq \frac{1}{n} \sum_v \left(\begin{array}{c} \bullet \\ \color{blue}{|} \\ \color{red}{|} \\ \color{yellow}{\square} v \end{array}, \begin{array}{c} \bullet \\ \color{red}{|} \\ \color{blue}{|} \\ \color{yellow}{\square} v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ \color{blue}{|} \\ \color{red}{|} \\ \color{yellow}{\square} v \end{array}, \begin{array}{c} \bullet \\ \color{red}{|} \\ \color{blue}{|} \\ \color{yellow}{\square} v \end{array} \right)^T$$

$$0 \leq a \begin{array}{c} \text{Blue Triangle} \\ \text{Blue Triangle} \end{array} + \frac{a+2c}{3} \begin{array}{c} \text{Red Triangle} \\ \text{Blue Triangle} \end{array} + \frac{b+2c}{3} \begin{array}{c} \text{Blue Triangle} \\ \text{Red Triangle} \end{array} + b \begin{array}{c} \text{Red Triangle} \\ \text{Red Triangle} \end{array}$$

$$\begin{array}{c}
 \text{Red Triangle} \\
 \text{Blue Triangle}
 \end{array}
 = 0 \begin{array}{c} \text{Blue Triangle} \\ \text{Red Triangle} \end{array} + 0 \begin{array}{c} \text{Red Triangle} \\ \text{Blue Triangle} \end{array} + 0 \begin{array}{c} \text{Blue Triangle} \\ \text{Red Triangle} \end{array} + \begin{array}{c} \text{Red Triangle} \\ \text{Red Triangle} \end{array}$$

$$\begin{array}{c} \text{Red Triangle} \\ \text{Red Triangle} \end{array}
 \geq \min\{0, 0, 0, 1\}$$

$$0 \geq -a \begin{array}{c} \text{Blue Triangle} \\ \text{Blue Triangle} \end{array} - \frac{a+2c}{3} \begin{array}{c} \text{Red Triangle} \\ \text{Blue Triangle} \end{array} - \frac{b+2c}{3} \begin{array}{c} \text{Blue Triangle} \\ \text{Red Triangle} \end{array} - b \begin{array}{c} \text{Red Triangle} \\ \text{Red Triangle} \end{array}$$

$$0 \geq d \left(0.6 \begin{array}{c} \text{Blue Triangle} \\ \text{Blue Triangle} \end{array} + \left(0.6 - \frac{1}{3} \right) \begin{array}{c} \text{Red Triangle} \\ \text{Blue Triangle} \end{array} + \left(0.6 - \frac{2}{3} \right) \begin{array}{c} \text{Blue Triangle} \\ \text{Red Triangle} \end{array} - 0.4 \begin{array}{c} \text{Red Triangle} \\ \text{Red Triangle} \end{array} \right)$$


$$\begin{pmatrix} a & c & 0 \\ c & b & 0 \\ 0 & 0 & d \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

$$\begin{array}{c} \bullet \\ \diagup \text{ (red)} \\ \bullet \\ \diagdown \text{ (red)} \\ \bullet \end{array} = 0 \begin{array}{c} \bullet \\ \diagup \text{ (blue)} \\ \bullet \\ \diagdown \text{ (blue)} \\ \bullet \end{array} + 0 \begin{array}{c} \bullet \\ \diagup \text{ (blue)} \\ \bullet \\ \diagdown \text{ (red)} \\ \bullet \end{array} + 0 \begin{array}{c} \bullet \\ \diagup \text{ (red)} \\ \bullet \\ \diagdown \text{ (blue)} \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagup \text{ (red)} \\ \bullet \\ \diagdown \text{ (red)} \\ \bullet \end{array}$$


$$0 \geq -a \begin{array}{c} \bullet \\ \diagup \text{ (blue)} \\ \bullet \\ \diagdown \text{ (blue)} \\ \bullet \end{array} - \frac{a+2c}{3} \begin{array}{c} \bullet \\ \diagup \text{ (blue)} \\ \bullet \\ \diagdown \text{ (red)} \\ \bullet \end{array} - \frac{b+2c}{3} \begin{array}{c} \bullet \\ \diagup \text{ (red)} \\ \bullet \\ \diagdown \text{ (blue)} \\ \bullet \end{array} - b \begin{array}{c} \bullet \\ \diagup \text{ (red)} \\ \bullet \\ \diagdown \text{ (red)} \\ \bullet \end{array}$$

$$0 \geq d \left(0.6 \begin{array}{c} \bullet \\ \diagup \text{ (blue)} \\ \bullet \\ \diagdown \text{ (blue)} \\ \bullet \end{array} + \left(0.6 - \frac{1}{3} \right) \begin{array}{c} \bullet \\ \diagup \text{ (blue)} \\ \bullet \\ \diagdown \text{ (red)} \\ \bullet \end{array} + \left(0.6 - \frac{2}{3} \right) \begin{array}{c} \bullet \\ \diagup \text{ (red)} \\ \bullet \\ \diagdown \text{ (blue)} \\ \bullet \end{array} - 0.4 \begin{array}{c} \bullet \\ \diagup \text{ (red)} \\ \bullet \\ \diagdown \text{ (red)} \\ \bullet \end{array} \right)$$

$$\begin{array}{c} \bullet \\ \diagup \text{ (red)} \\ \bullet \\ \diagdown \text{ (red)} \\ \bullet \end{array} \geq \min \left\{ 0.6d - a, \left(0.6 - \frac{1}{3} \right) d - \frac{a+2c}{3}, \right. \\ \left. \left(0.6 - \frac{2}{3} \right) d - \frac{b+2c}{3}, 1 - 0.4d - b \right\}$$



$$\geq 0.14150099\dots$$



$$\geq \min \left\{ 0.6d - a, \left(0.6 - \frac{1}{3} \right) d - \frac{a + 2c}{3}, \right. \\ \left. \left(0.6 - \frac{2}{3} \right) d - \frac{b + 2c}{3}, 1 - 0.4d - b \right\}$$

Numerical solution from CSDP:

$$a = 6 \times 0.1200006508849779385$$

$$a = 0.72$$

$$b = 6 \times 0.05333290843810910981$$


$$b = 0.32$$

$$c = 6 \times -0.07999989818128358521$$

$$c = -0.48$$


$$d = 1.400006454027185265$$

$$d = 1.4$$

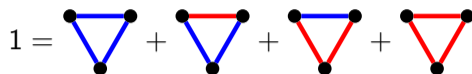


$$\geq \min \{ 0.12, 0.45\bar{3}, 0.12, 0.12 \} = 0.12$$

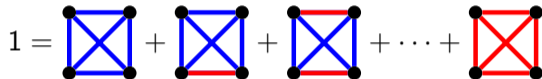
HOW TO IMPROVE 0.12?

 $\geq 0.14150099\dots$


Sample bigger graphs. Instead of

$$1 = \text{triangle} + \text{triangle} + \text{triangle} + \text{triangle}$$


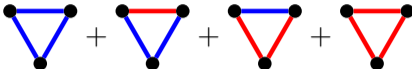
use

$$1 = \text{square} + \text{square} + \text{square} + \dots + \text{square}$$


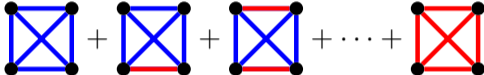
HOW TO IMPROVE 0.12?


 $\geq 0.14150099\dots$

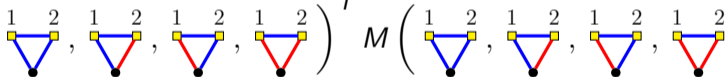
Sample bigger graphs. Instead of

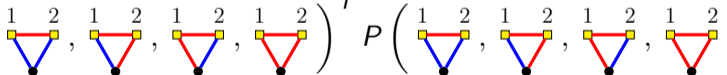
$$1 = \text{triangle}_1 + \text{triangle}_2 + \text{triangle}_3 + \text{triangle}_4$$


use


$$1 = \text{square}_1 + \text{square}_2 + \text{square}_3 + \dots + \text{square}_n$$


and include also $M, P \succcurlyeq 0$

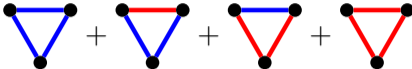
$$0 \leq \begin{pmatrix} \text{triangle}_1 & \text{triangle}_2 & \text{triangle}_3 & \text{triangle}_4 \end{pmatrix}^T M \begin{pmatrix} \text{triangle}_1 & \text{triangle}_2 & \text{triangle}_3 & \text{triangle}_4 \end{pmatrix}$$


$$0 \leq \begin{pmatrix} \text{triangle}_1 & \text{triangle}_2 & \text{triangle}_3 & \text{triangle}_4 \end{pmatrix}^T P \begin{pmatrix} \text{triangle}_1 & \text{triangle}_2 & \text{triangle}_3 & \text{triangle}_4 \end{pmatrix}$$


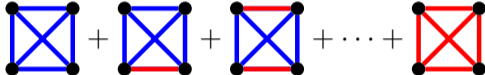
HOW TO IMPROVE 0.12?


 $\geq 0.14150099\dots$

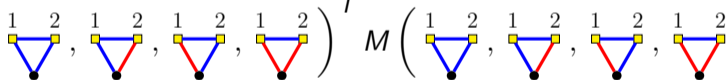
Sample bigger graphs. Instead of

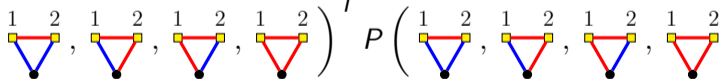
$$1 = \text{triangle}_1 + \text{triangle}_2 + \text{triangle}_3 + \text{triangle}_4$$


use


$$1 = \text{square}_1 + \text{square}_2 + \text{square}_3 + \dots + \text{square}_n$$


and include also $M, P \succcurlyeq 0$

$$0 \leq \begin{pmatrix} \text{triangle}_1 & \text{triangle}_2 & \text{triangle}_3 & \text{triangle}_4 \end{pmatrix}^T M \begin{pmatrix} \text{triangle}_1 & \text{triangle}_2 & \text{triangle}_3 & \text{triangle}_4 \end{pmatrix}$$


$$0 \leq \begin{pmatrix} \text{triangle}_1 & \text{triangle}_2 & \text{triangle}_3 & \text{triangle}_4 \end{pmatrix}^T P \begin{pmatrix} \text{triangle}_1 & \text{triangle}_2 & \text{triangle}_3 & \text{triangle}_4 \end{pmatrix}$$


This gives


 $\geq 0.127815\dots$

HOW TO IMPROVE $0.12781\dots$?

Sample even bigger graphs.


Use K_5 instead of K_4

Include even more types and flags.



$\geq 0.14150099\dots$

HOW TO IMPROVE 0.12781...?


 $\geq 0.14150099\dots$

Sample even bigger graphs.


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This gives

 $\geq 0.1333333 = 2/15.$

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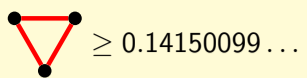
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Try even bigger!

HOW TO IMPROVE 0.12781...?

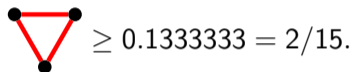


Sample even bigger graphs.

Use K_5 instead of K_4

Include even more types and flags.

This gives




Try even bigger!

vertices	# graphs	time	bound
3	4	instant	0.12
4	11	instant	0.127815...
5	34	instant	0.13333...
6	156	seconds	0.13333...
7	1044	minutes	0.13333...
8	12346	day(s)	0.13333...
9	274668	not computable*	?

* needs hundreds of GB of RAM, maybe easy in 10 years?

GETTING 0.14150099...


$$\geq 0.14150099\dots \text{ for } p = 0.6 \text{ by Razborov}$$


$$\left\{ \begin{array}{l} \text{Minimize} \\ \text{subject to} \end{array} \right. \quad \begin{array}{c} \img alt="A red triangle with three black vertices and three red edges." data-bbox="378 291 440 389"/> \\ \img alt="A red vertical edge with two black vertices and one red edge." data-bbox="451 394 465 500"/> \end{array} - p \geq 0$$

GETTING 0.14150099...

$$\text{triangle} \geq 0.14150099\dots \text{ for } p = 0.6 \text{ by Razborov}$$

$$\left\{ \begin{array}{l} \text{Minimize} \\ \text{subject to} \end{array} \right. \text{triangle} \cdot \left(\text{edge} - p \right) \geq 0 \text{ for any graph } H$$

GETTING 0.14150099...


 $\geq 0.14150099\dots$ for $p = 0.6$ by Razborov


$$\left\{ \begin{array}{l} \text{Minimize} \\ \text{subject to} \end{array} \right. \quad \begin{array}{c} \text{triangle} \\ H \cdot \left(\text{edge} - p \right) \geq 0 \text{ for any graph } H \end{array}$$

vertices	# graphs	time	bound	new bound
3	4	instant	0.12	0.12
4	11	instant	0.12781...	0.131746...
5	34	instant	0.13333...	0.14046241...
6	156	seconds	0.13333...	0.14150099...
7	1044	minutes	0.13333...	0.14150099...
8	12346	day(s)	0.13333...	0.14150099...

These are just **numerical** bounds! Not exact.


GOODMAN'S BOUND

Recall we got for $p = 0.6$


$$\geq \min \left\{ 0.6d - a, \left(0.6 - \frac{1}{3}\right) d - \frac{a + 2c}{3}, \left(0.6 - \frac{2}{3}\right) d - \frac{b + 2c}{3}, 1 - (1 - 0.6)d - b \right\}$$

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Same thing holds when 0.6 is replaced by a parameter p .

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$$a = 2p^2 \quad b = 2p^2 - 4p + 2 \quad c = p(2p - 2) \quad d = 4p - 1$$

gives Goodman's bound:


$$\geq 2p^2 - p$$

GOODMAN'S BOUND

Recall we got for $p = 0.6$


$$\geq \min \left\{ 0.6d - a, \left(0.6 - \frac{1}{3}\right) d - \frac{a + 2c}{3}, \left(0.6 - \frac{2}{3}\right) d - \frac{b + 2c}{3}, 1 - (1 - 0.6)d - b \right\}$$

Same thing holds when 0.6 is replaced by a parameter p .


$$a = 2p^2$$

$$b = 2p^2 - 4p + 2$$

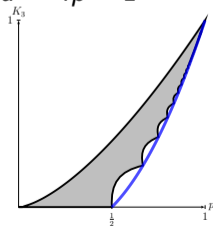
$$c = p(2p - 2)$$

$$d = 4p - 1$$

gives Goodman's bound:


$$\geq 2p^2 - p$$

This is tight for $p \in \{\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\}$.



Thank you!