Crossing numbers of complete bipartite graphs

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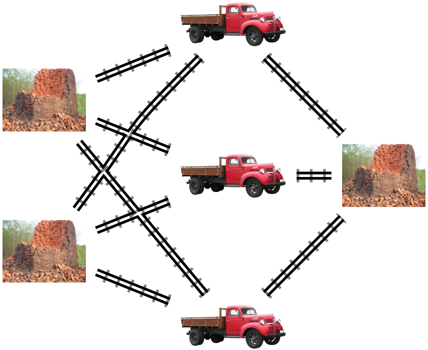
Crossing number

Turán 1945: In a forced labor camp, prisoners transfer carts of bricks from kilns to shipping yards.

When two tracks cross, cart is likely to derail.

How to connect every kiln and shipping yard that minimizes the number of crossings?



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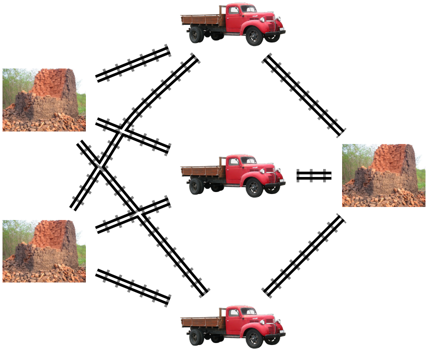

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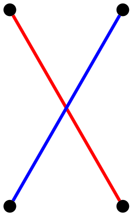
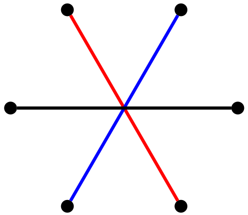
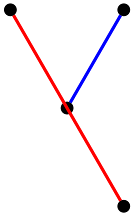
Allowed Drawings and Crossings



allowed



not allowed



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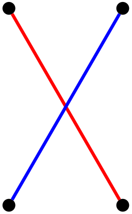

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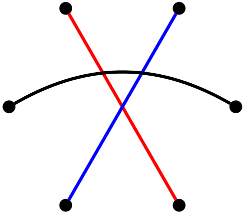

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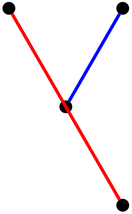


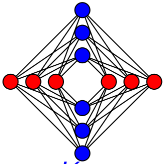
not allowed

Conjectures

For a graph G, the *crossing number* $cr(G)$ is the minimum number of crossings over all (proper) drawings of G in the plane.

Conjecture (Zarankiewicz 1954)

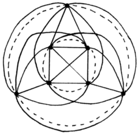
$$cr(K_{m,n}) = Z(m,n) := \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{(n-1)}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{(m-1)}{2} \right\rfloor .$$



$K_{6,6}$

Conjecture (Hill 1963)

$$cr(K_n) = H(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$



Conjectures

For a graph G, $cr(G)$ is the *crossing number*.

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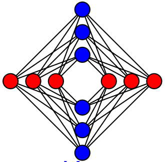

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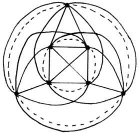


Theorem (Kainen, 1968; Richter and Thomassen, 1993)

$$\lim_{n\to\infty} \frac{cr(K_{n,n})}{Z(n,n)} = 1 \implies \lim_{n\to\infty} \frac{cr(K_n)}{H(n)} = 1$$

For a graph G, $cr(G)$ is the *crossing number*.

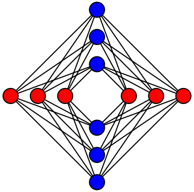
CONJECTURE (ZARANKIEWICZ 1954)

$$cr(K_{m,n}) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{(n-1)}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{(m-1)}{2} \right\rfloor.$$

THEOREM (NORIN, ZWOLS 2013)

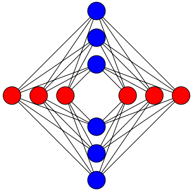
$$cr(K_{m,n}) \geq 0.9 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{(n-1)}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{(m-1)}{2} \right\rfloor$$

for large m and n.
(Zarankiewicz's conjecture is 90% true)



$K_{6,6}$

80% Kleitman 1970
83% De Klerk, Maharry, Pasechnik, Richter, Salazar 2006
85.9% De Klerk, Pasechnik, Schrijver 2007



$K_{6,6}$

THEOREM (NORIN, ZWOLS 2013)

$$cr(K_{m,n}) \geq 0.905\ Z(m, n)$$

for large m and n.

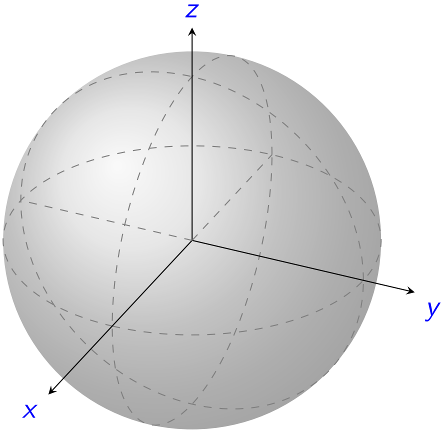
THEOREM (BALOGH, L., NORIN, PFENDER, SALAZAR, SPIRO 2023)

$$cr(K_{m,n}) \geq 0.91198\ Z(m, n)$$

for large m and n.

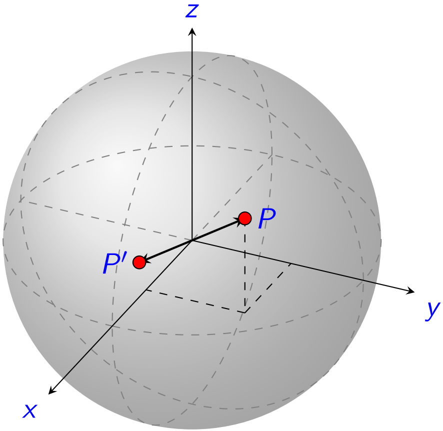
Same method of Flag Algebras, pushed a little bit more.

Many Different Constructions for $Z(m, n)$



- Take $m/2 + n/2$ pairs of antipodal points
- Connect points with geodesics
- Get $Z(m, n)$ crossings

Many Different Constructions for $Z(m, n)$



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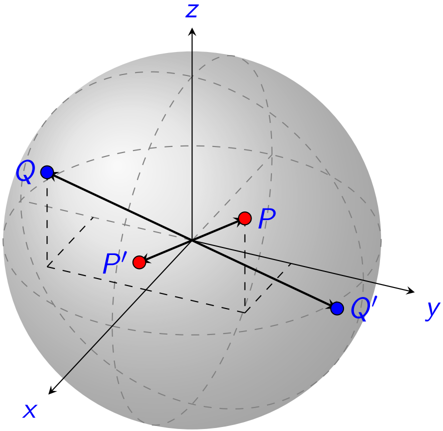

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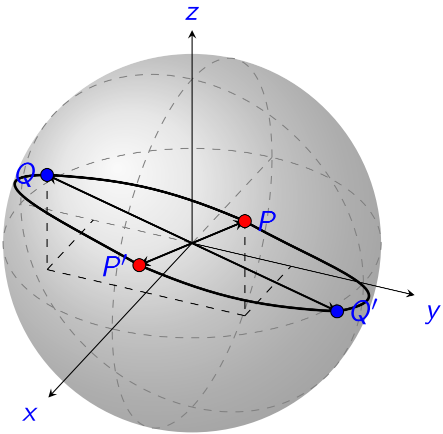

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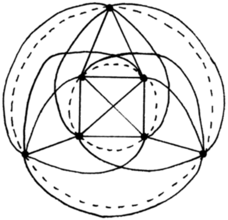
Hill's Conjecture

$cr(K_{n,n}) \geq 0.91198\ Z(n,n)$ implies $cr(K_n) \geq 0.91198\ H(n)$

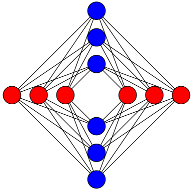


Theorem (Balogh, L., Salazar 2019)

$cr(K_n) \geq 0.985 H(n)$, *i.e. Hill is 98.5% true for large* n.

Restricting drawing of $K_{m,n}$

Rectilinear crossing number is with straight line drawing.



$K_{6,6}$

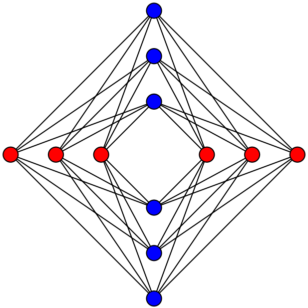
Theorem (Balogh, L., Norin, Pfender, Salazar, Spiro 2023)
Rectilinear version of the Zarankiewicz is 98.7% true for large m and n.

Rectilinear K_n is 99.8% known, connected to Sylvester Four Point Problem.

More restrictions on drawing

- Rectilinear drawing, where red and blue vertices are separated by a straight line.
  99.5%

- Rectilinear drawing, where red and blue vertices are separated by a straight line, all vertices on a curve not crossed by any edge (2-page drawing).
  99.998%



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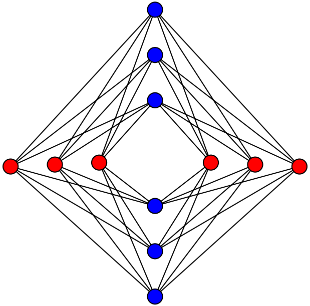

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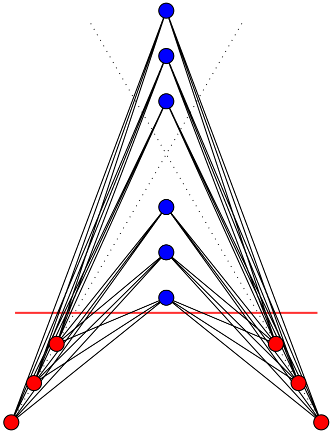

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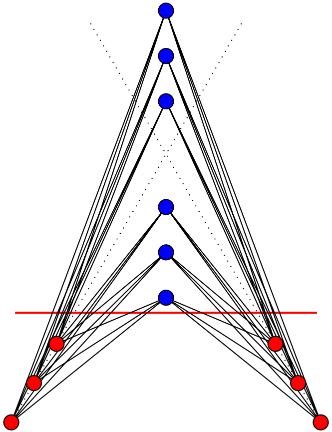
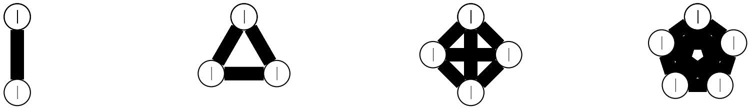


Turán type questions

Theorem (Turán 1941)

K_q-free graph on n vertices maximizing the number of edges is $T_{q-1}(n)$.



$T_2(n)$ $T_3(n)$ $T_4(n)$ $T_5(n)$

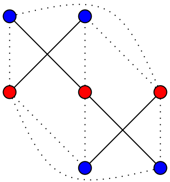
Problem (Generalized Turán)

In an A-free structure on n vertices, max/min the number of copies of structure B.

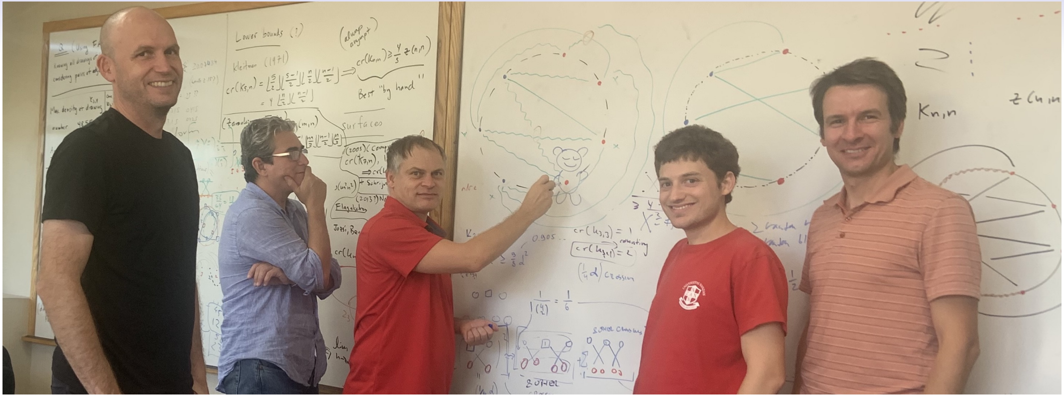
Turán Type Result

Theorem (Balogh, L., Norin, Pfender, Salazar, Spiro)

If a drawing of $K_{n,n}$ has no $K_{3,4}$ with exactly two crossings sharing one vertex, then it has at last $n^4/16 + o(n^4)$ crossings.



i.e. Zarankiewicz is $100\% - o(1)$ true



Floran Pfender, Gelasio Salazar, József Balogh, Sam Spiro, Bernard Lidický

and remotely Sergey Norin 

American Institute of Mathematics

Method of Flag Algebras

Seminal paper:
Razborov, Flag Algebras, *Journal of Symbolic Logic* **72** (2007), 1239–1282.
David P. Robbins Prize by AMS for Razborov in 2013

Example

If density of edges in a graph is p, what is the minimum density of triangles?

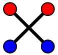
- Designed to attack extremal problems.
- Works well if constraints as well as desired value can be computed by checking small subgraphs (or average over small subgraphs).
- The results are for the limit as graphs get very large.
- Applicable to graphs, hypergraphs, oriented graphs, phylogenetic trees,. . . crossings

Idea in our setting

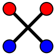
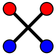
Density of F in G

$$d(F,G) := \frac{\#\text{ of subgraphs of } G \text{ isomorphic to } F}{\binom{n}{|V(F)|}} \in [0,1]$$

Let G be a drawing of $K_{n,n}$. Want to show

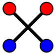
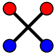
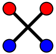
$d\Big($  $, G\Big) \geq \textit{NiceBound}$

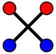
Denote all drawings of $K_{3,4}$ (up to isomprhism) by $\mathcal{K}_{3,4}$

$d\Big($  $, G\Big) = \sum_{H \in \mathcal{K}_{3,4}} d\Big($  $, H\Big)\, d(H,G).$

Idea in our setting

Denote all drawings of $K_{3,4}$ (up to isomprhism) by $\mathcal{K}_{3,4}$.

$d\Big($  $, G\Big) = \sum_{H \in \mathcal{K}_{3,4}} d\Big($  $, H\Big) d(H, G) \geq \min_{H \in \mathcal{K}_{3,4}} d\Big($  $, H\Big) = BadBound > 0$

since $cr(K_{3,4}) \geq 2$, $d\Big($  $, H\Big) > 0$ for all H.

Power of Flag Algebras:
Find coefficients c_H, where for all G as $|V(G)| \to \infty$.

$$\sum_{H \in \mathcal{K}_{3,4}} c_H d(H, G) \geq 0 + o(1)$$

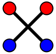
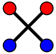
Interesting if $c_H < 0$ for some H.

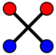
Using extra inequlities

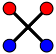
If we have

$$\sum_{H \in \mathcal{K}_{3,4}} c_H d(H, G) \geq 0 + o(1)$$

then

$d\Big($  $, G\Big) = \sum_{H \in \mathcal{K}_{3,4}} d\Big($  $, H\Big) d(H, G)$

$\geq \sum_{H \in \mathcal{K}_{3,4}} d\Big($  $, H\Big) d(H, G) - \sum_{H \in \mathcal{K}_{3,4}} c_H d(H, G)$

$\geq \min_{H \in \mathcal{K}_{3,4}} \Big\{ d\Big($  $, H\Big) - c_H \Big\} = \mathit{MaybeNiceBound}$

Recall maybe some $c_H < 0$.

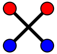
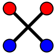
Obtaining ≥ 0 for all G

For all $c_F \in \mathbb{R}$

$$0 \leq \left(\sum_{F \in \mathcal{K}_{k,\ell}} c_F \, d(F, G) \right)^2$$

Flag Algebras can expand $(\ldots)^2$ as a linear combination $+o(1)$.
Optimizing a Sum of Squares problem

Solving

$d($  $, G) \geq \max_{c_H} \min_{H \in \mathcal{K}_{3,4}} \{ d($  $, H) - c_H \}$

can be done using semidefinite programming solver.

Semidefinite and Linear programming

Linear program (LP)

$$(LP) \begin{cases} \text{minimize} & 3x_1 + x_2 + x_3 \\ \text{subject to} & x_1 + 2x_2 + 3x_3 \leq 6 \\ & 4x_2 + x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0 \end{cases}$$

Semidefinite program (SDP)

$$(SDP) \begin{cases} \text{minimize} & 3x_1 + x_2 + x_3 \\ \text{subject to} & x_1 + 2x_2 + 3x_3 \leq 6 \\ & 4x_2 + x_3 \leq 4 \\ & \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \succeq 0 \end{cases}$$

(SDP) efficiently numerically solvable using solver CSDP, SDPA, MOSEK

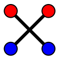
Each $H \in \mathcal{K}_{3,4}$ corresponds to one constraints in (SDP).

Up to 200,000 constraints solvable (with patience).

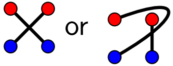
Isomorphism

When are two drawings isomorphic?

- Does a pair of edges induce a crossing?



- Do four vertices induce a crossing?



- For each vertex, cyclic order of neighbors (rectilinear)

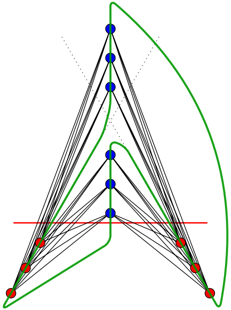
More information gives larger $|\mathcal{K}_{r,\ell}|$ (maybe better bounds).

But larger r, ℓ allows for more squares (maybe better bounds).

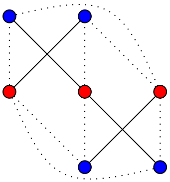
Sizes of calculations

- $cr(K_{n,n})$: pairs of crossing edges, $|\mathcal{K}_{3,4}| = 6,393$
- rectilinear: pairs of crossing edges, $|\mathcal{K}_{4,4}| = 231,922$
- Turán result: quadruples inducing crossing, $|\mathcal{K}_{3,4}| = 355$

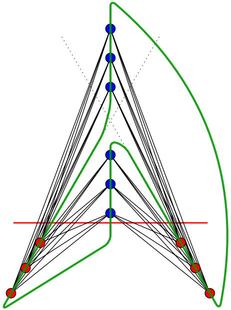
Summary



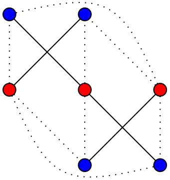
- Flag Algebras is applicable to different variants of the crossing number.
- Determining $cr(K_{n,n})$ seems to be hard.
- It is possible to get a Turán type result.



Summary



- Flag Algebras is applicable to different variants of the crossing number.
- Determining $cr(K_{n,n})$ seems to be hard.
- It is possible to get a Turán type result.



Gracias Thank You