Flag Algebras and Its Application

Bernard Lidický

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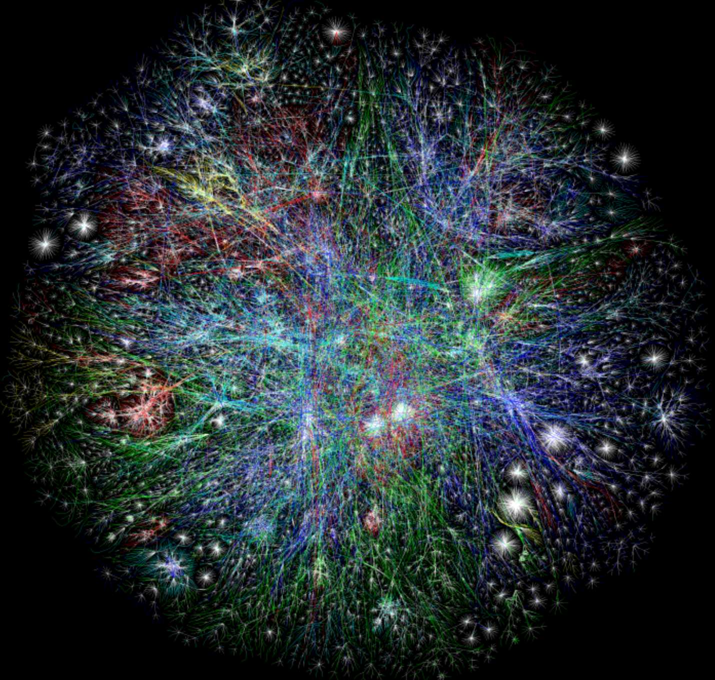
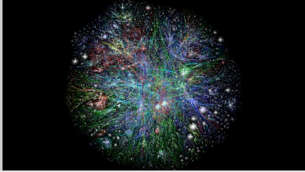




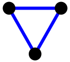
We will only consider large graphs (or networks).

Here is a graph of the internet from a while back to show there are large graphs that are interesting.

Inspirational Problem

- Let n be a fixed number of vertices in a graph G.
- Assume G has m edges.
- What is the number of triangles in G?

edge is 

triangle is 

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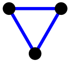
triangle is 

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Liu, Pikhurko, Staden 2020 (144 pages)

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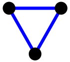
triangle is 

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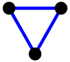
triangle is 

Consider $n \to \infty$.
Edges $= p\binom{n}{2}$
Triangles $= t\binom{n}{3}$
Now $p, t \in [0, 1]$.

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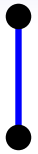
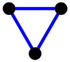
edge is 

triangle is 

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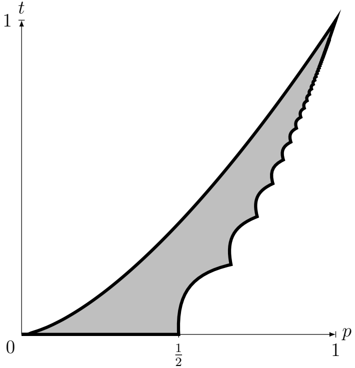
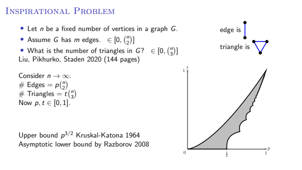


Upper bound $p^{3/2}$ Kruskal-Katona 1964
Asymptotic lower bound by Razborov 2008



The edge (0,0) to (1/2,0) is Mantel's theorem. The sharp points are Turán graphs. The figure is exaggerated to show the idea if the shape.

Flag algebras

Seminal paper:
Razborov, Flag Algebras, *Journal of Symbolic Logic* **72** (2007), 1239–1282.
David P. Robbins Prize by AMS for Razborov in 2013 over 300 citations (on Google)

Example

If density of edges is p, what is the minimum density of triangles?

- Designed to attack extremal problems.
- Works well if constraints as well as desired value can be computed by checking small subgraphs (or average over small subgraphs).
- The results are for the limit as graphs get very large.

Example extremal problem

Theorem (Mantel 1907)

Every n-vertex triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

Problem

Maximize a graph parameter (# of edges) over a class of graphs (triangle-free).

- local condition and global parameter (computable locally)
- threshold
- bound and extremal example

Proof of Mantel's Theorem

Theorem (Mantel 1907)

In every n-vertex triangle-free graph $|E| \le \frac{1}{4}n^2$.

Proof.

$$n|E| \ge \sum_{ij\in E} \underbrace{(d_i + d_j)}_{\le n} = \sum_{i\in V} d_i^2 \ge \frac{(\sum_{i\in V} d_i)^2}{n} = \frac{4|E|^2}{n}$$

□

Cauchy-Schwarz $(\sum_i a_i b_i)^2 \le \sum_i a_i^2 \cdot \sum_i b_i^2$ with $b_i = 1$.
Cauchy-Schwarz $(\sum_i a_i 1)^2 \le \sum_i a_i^2 \cdot \sum_i 1^2$.

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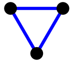
$|E|$ is number of edges
d_i is a degree of a vertex i. $d_i + d_j \le n$ because the graph is triangle-free.
We will try to rewrite the proof using densities and this should get us familiar with flag algebras notation.

Flag algebras definitions

Let G be a graph on n vertices.



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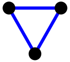

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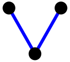

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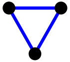

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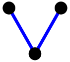

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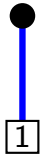
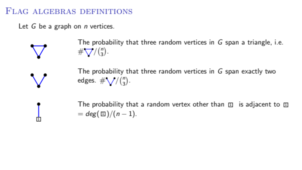


The probability that a random vertex other than $\boxed{1}$ is adjacent to $\boxed{1}$ $= deg(\boxed{1})/(n-1)$.



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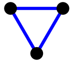

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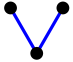

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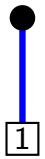

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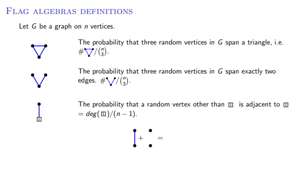

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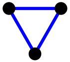

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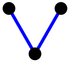

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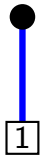

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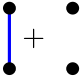
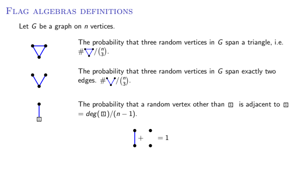
 $= 1$



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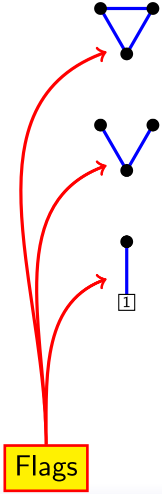

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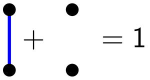

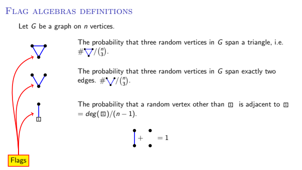

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In every n-vertex triangle-free graph $|E| \leq \frac{1}{4}n^2$.

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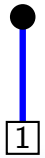
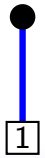
$|E| =$  $\binom{n}{2} \approx$  $\frac{n^2}{2}$, $\quad d_1 =$  $(n-1) \approx$  $\cdot\, n$, $\quad 1 = \boxed{1}$

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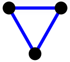
In every n-vertex triangle-free graph $|E| \leq \frac{1}{4}n^2$. *If*  $= 0$ *then*  $\leq \frac{1}{2}$

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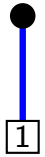
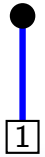
$|E| =$  $\binom{n}{2} \approx$  $\frac{n^2}{2}$, $\quad d_1 =$  $(n-1) \approx$  $\cdot\, n$, $\quad 1 = \boxed{1}$

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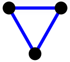
In every n-vertex triangle-free graph $|E| \leq \frac{1}{4}n^2$. *If*  $= 0$ *then*  $\leq \frac{1}{2}$

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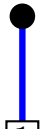
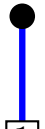
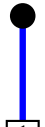
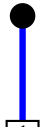
$n^2 \times$  $\frac{n^2}{2} \geq \sum_{\boxed{1} \in V} \boxed{1}^2 \cdot \sum_{\boxed{1} \in V} \Big($  $n\Big)^2 \geq \Big(\sum_{\boxed{1} \in V} \boxed{1} \cdot$  $n\Big)^2 = 4\Big($  $\frac{n^2}{2}\Big)^2$

$|E| =$  $\binom{n}{2} \approx$  $\frac{n^2}{2}$, $\quad d_1 =$  $(n-1) \approx$  $\cdot\, n$, $\quad 1 = \boxed{1}$

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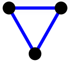
In every n-vertex triangle-free graph $|E| \leq \frac{1}{4}n^2$. *If*  $= 0$ *then*  $\leq \frac{1}{2}$

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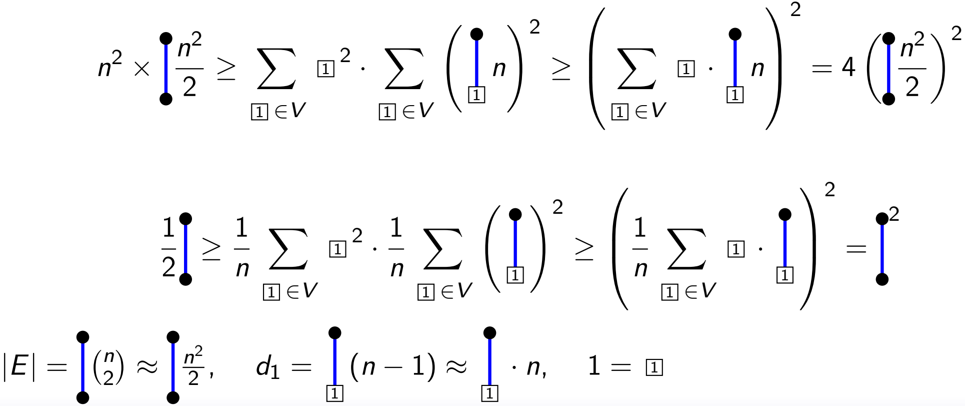

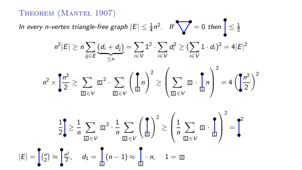

We are ignoring lower order terms and approximate $\binom{n}{k}$ by $\frac{n^k}{k!}$.

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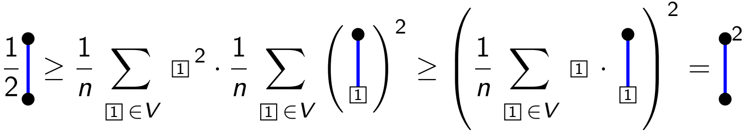

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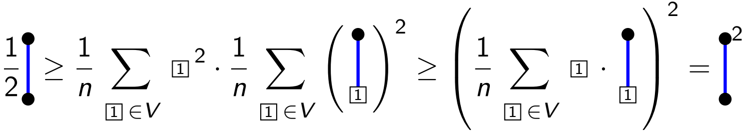

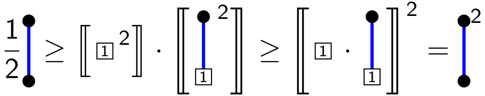

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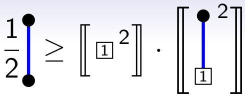
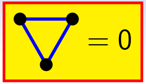
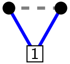
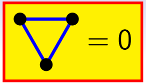


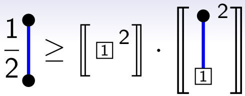


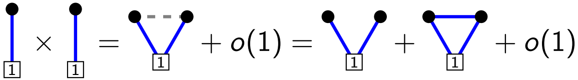
$\boxed{1}$ = not choosing anything = 1

 = probability of choosing a vertex . . . $deg(\boxed{1})/(n-1)$

 = probability of choosing two distinct vertices . . . $\binom{deg(\boxed{1})}{2} / \binom{n-1}{2}$







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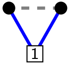
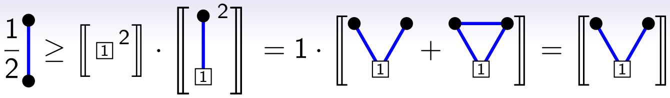
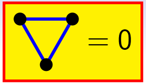
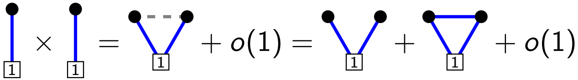
 = probability of choosing two distinct vertices ... $\binom{deg(\boxed{1})}{2}/\binom{n-1}{2}$

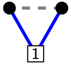


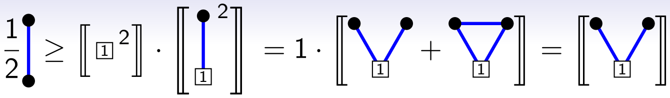


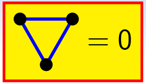


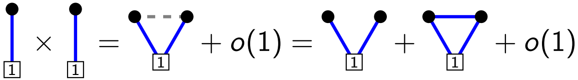
$\boxed{1}$ = not choosing anything = 1

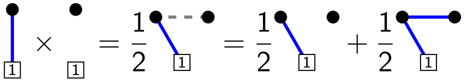
 = probability of choosing a vertex … $deg(\boxed{1})/(n-1)$

 = probability of choosing two distinct vertices … $\binom{deg(\boxed{1})}{2}/\binom{n-1}{2}$

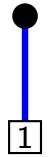


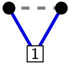


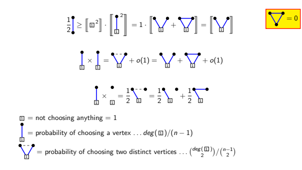




$\boxed{1}$ = not choosing anything = 1

 = probability of choosing a vertex . . . $deg(\boxed{1})/(n-1)$

 = probability of choosing two distinct vertices . . . $\binom{deg(\boxed{1})}{2} / \binom{n-1}{2}$

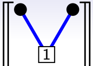


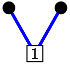
Probability of choosing a vertex inducing an edge with the fixed vertex 1.

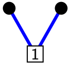
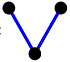
Probability of choosing a pair of distinct vertices each being in an edge with the fixed vertex 1.

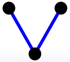
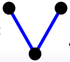
Notice on the left each pair of vertices counted twice!

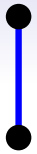
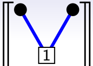
We will ignore $o(1)$ in the future

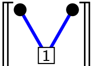
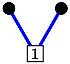
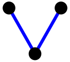
$\frac{1}{2}$  $\geq [\![$  $]\!]$

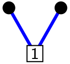
 $=$ probability of choosing two distinct vertices . . . $\binom{deg(\boxed{1})}{2} / \binom{n-1}{2}$

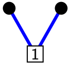
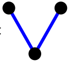
$\sum_{\boxed{1} \in V}$  $\binom{n-1}{2} = \#$ 

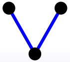
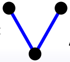
 $=$ probability of choosing three distinct vertices . . . $\#$  $/ \binom{n}{3}$

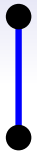
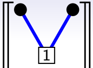
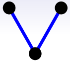
$\frac{1}{2}$  $\geq \llbracket$  $\rrbracket$

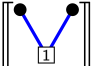
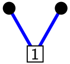
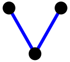
$\llbracket$  $\rrbracket = \frac{1}{n} \sum_{\boxed{1} \in V}$  $= \frac{1}{3}$ 

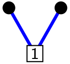
 $=$ probability of choosing two distinct vertices . . . $\binom{deg(\boxed{1})}{2} / \binom{n-1}{2}$

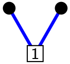
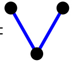
$\sum_{\boxed{1} \in V}$  $\binom{n-1}{2} = \#$ 

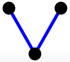
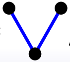
 $=$ probability of choosing three distinct vertices . . . $\#$  $/ \binom{n}{3}$

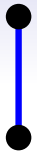
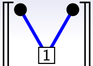
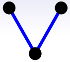
$\frac{1}{2}$  $\geq \left[\!\left[$  $\right]\!\right] = \frac{1}{3}$ 

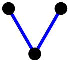
$\left[\!\left[$  $\right]\!\right] = \frac{1}{n} \sum_{\boxed{1} \in V}$  $= \frac{1}{3}$ 

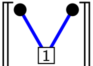
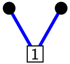
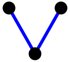
 = probability of choosing two distinct vertices . . . $\binom{deg(\boxed{1})}{2} / \binom{n-1}{2}$

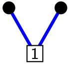
$\sum_{\boxed{1} \in V}$  $\binom{n-1}{2} = \#$ 

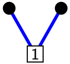
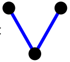
 = probability of choosing three distinct vertices . . . $\#$  $/ \binom{n}{3}$

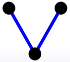
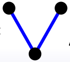
$\frac{1}{2}$  $\geq \Big[\!\Big[$  $\Big]\!\Big] = \frac{1}{3}$ 

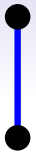
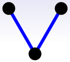
 $\geq \frac{2}{3}$ 

$\Big[\!\Big[$  $\Big]\!\Big] = \frac{1}{n} \sum_{\boxed{1} \in V}$  $= \frac{1}{3}$ 

 = probability of choosing two distinct vertices ... $\binom{deg(\boxed{1})}{2} / \binom{n-1}{2}$

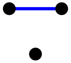
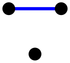
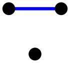
$\sum_{\boxed{1} \in V}$  $\binom{n-1}{2} = \#$ 

 = probability of choosing three distinct vertices ... $\#$  $/ \binom{n}{3}$

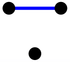
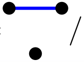
 $\geq \frac{2}{3}$ 

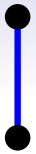
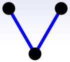
$$|E| = \frac{1}{n}\left(\sum 1 + \sum 2 + \sum 3\right)$$

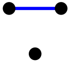
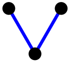
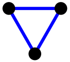
(summation subscripts:)   

 $\binom{n}{2} \approx \frac{1}{n}$  $\binom{n}{3} + \frac{2}{n}$  $\binom{n}{3} + \frac{3}{n}$  $\binom{n}{3}$

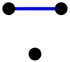
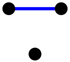
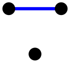
 = probability of choosing an edge . . . $|E|/\binom{n}{2}$

 = probability of choosing an triple . . . #  $/\binom{n}{2}$

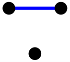
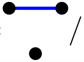
 $\geq \frac{2}{3}$ 

 $= \frac{1}{3}$  $+ \frac{2}{3}$  $+ \frac{3}{3}$ 

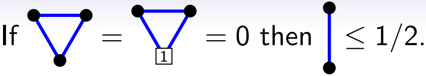
$|E| = \frac{1}{n} \Big(\sum$  $1 + \sum$  $2 + \sum$  $3 \Big)$

 $\binom{n}{2} \approx \frac{1}{n}$  $\binom{n}{3} + \frac{2}{n}$  $\binom{n}{3} + \frac{3}{n}$  $\binom{n}{3}$

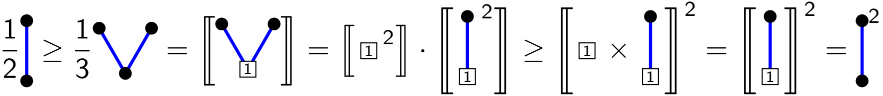
 = probability of choosing an edge ... $|E| / \binom{n}{2}$

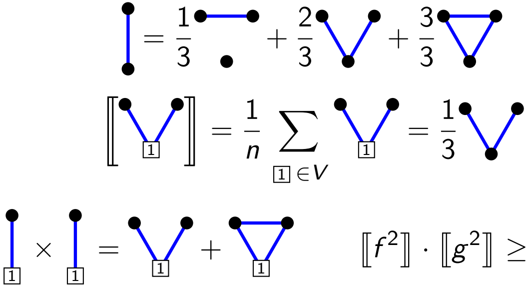
 = probability of choosing an triple ... #  $/ \binom{n}{2}$

Proof recap

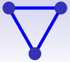
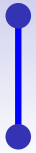


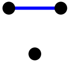
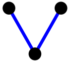
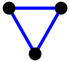
$$n^2|E| \geq \sum_{i\in V} 1^2 \cdot \sum_{i\in V} d_i^2 \geq \left(\sum_{i\in V} 1 \cdot d_i\right)^2 = 4|E|^2$$

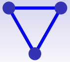
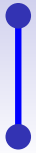


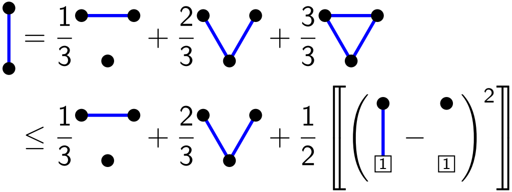


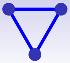
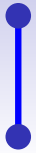
$$[\![f^2]\!] \cdot [\![g^2]\!] \geq [\![f \cdot g]\!]^2$$

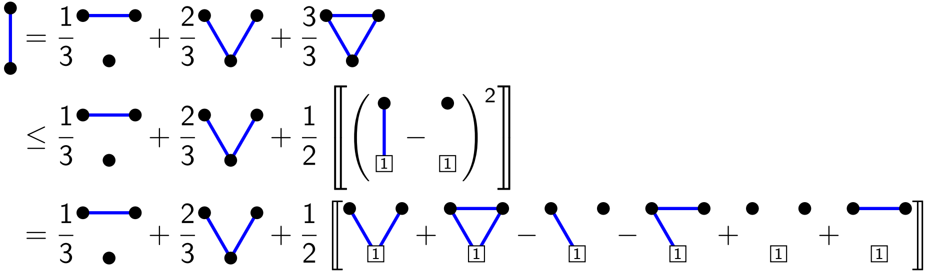
Different proof of  $= 0$ implies  $\leq 1/2$

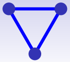
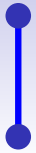
 $= \frac{1}{3}$  $+ \frac{2}{3}$  $+ \frac{3}{3}$ 

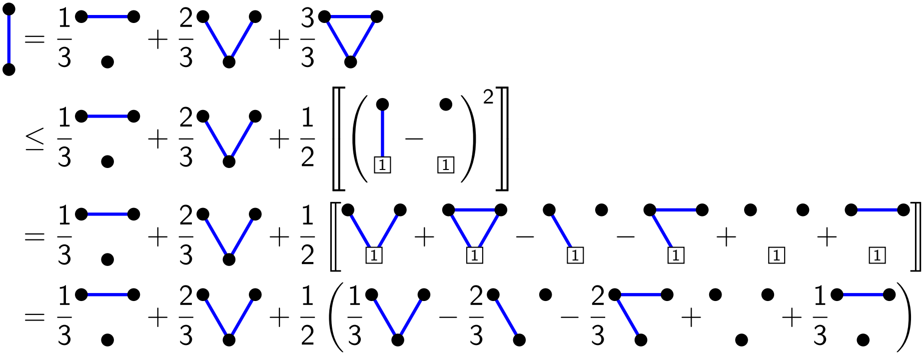
Different proof of  = 0 implies  ≤ 1/2

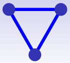
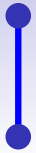


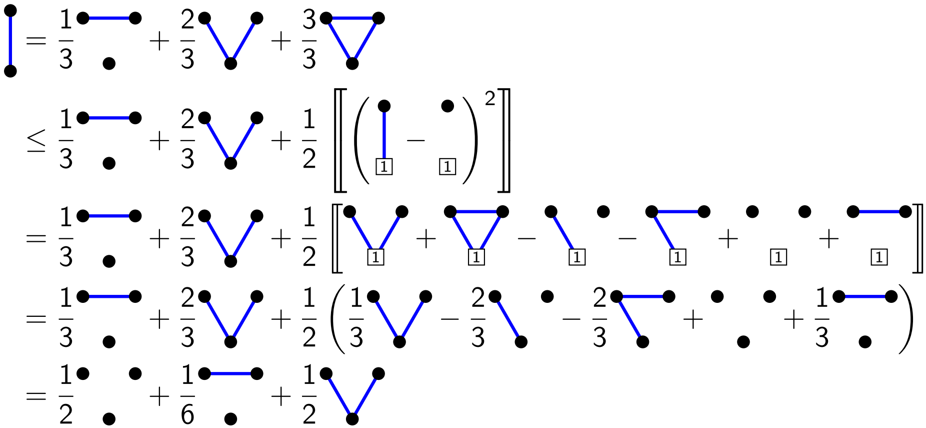
Different proof of  = 0 implies  ≤ 1/2

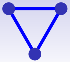


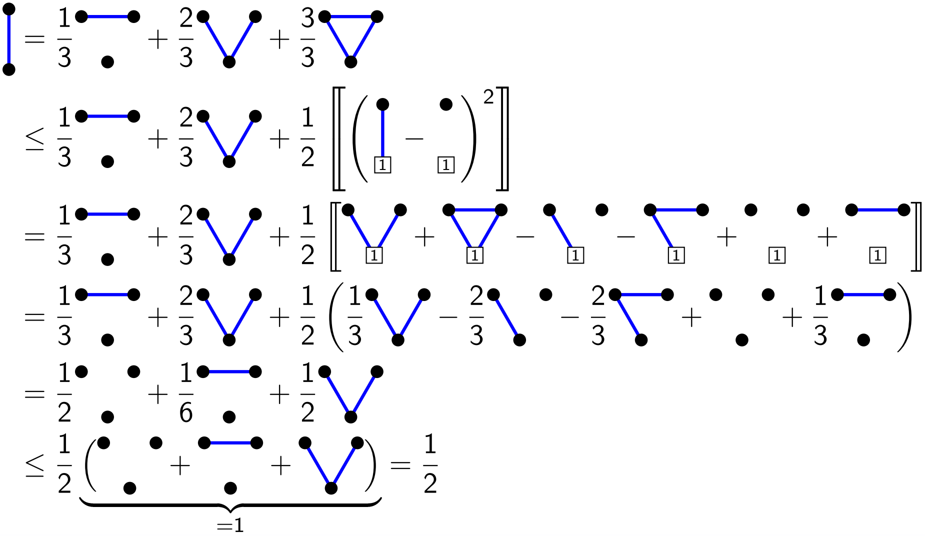
Different proof of  = 0 implies  ≤ 1/2



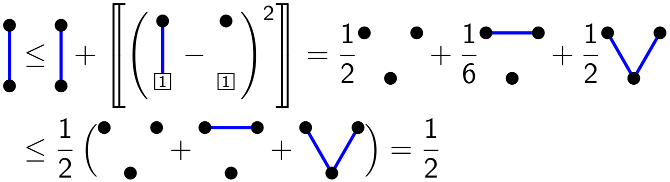
Different proof of  $= 0$ implies  $\leq 1/2$



Different proof of  = 0 implies  ≤ 1/2



Automated approach

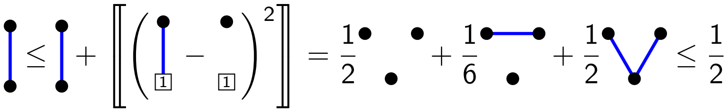


In general as sum of squares

$$f \leq f + \sum_{h} [\![h^2]\!] = \sum_{G \in \mathcal{F}_n} c_G \cdot G \leq \max_{G \in \mathcal{F}_n} c_G$$

f, g linear combination of flags
$\mathcal{F}_n$ ... flags on n vertices
SOS proofs can be optimized by semidefinite programming

Sum of squares



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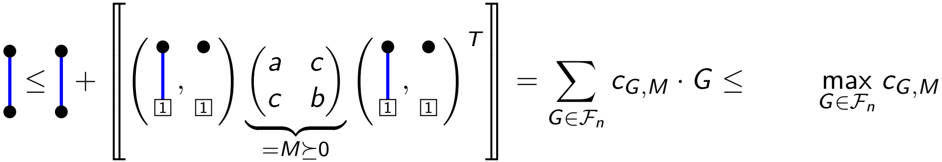
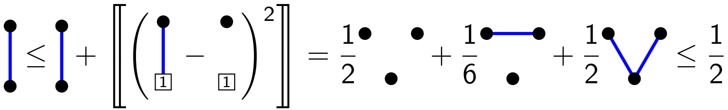
Semidefinite matrix



Sum of squares



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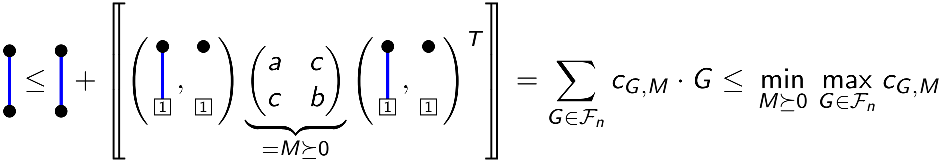

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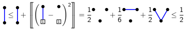

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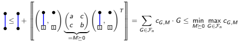

Make the subscript of Max under

Rainbow Triangles

Rainbow Triangles

https://arxiv.org/abs/2511.21061

Joints

Joint in $\mathbb{R}^d$ is a point where d lines that span $\mathbb{R}^d$ intersect.
What is the maximum number of joints for N lines?

Theorem (Chao and Hans Yu 2023+)

Number of joints is maximized by k hyperplanes whose intersection give $N = \binom{k}{d-1}$ lines and $\binom{k}{d}$ joints.

Assymptotically by Hans Yu and Zhao 2023.

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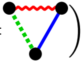
Multijoint problem: In $\mathbb{R}^3$, lines of three colors, maximize rainbow joints.
hyperplane $\rightarrow$ vertex
intersection of hyperplanes $\rightarrow$ edge
joint $\rightarrow$ rainbow triangle

Theorem (Chao and Hans Yu 2024+)

In 3-edge colored graph $\Big(\#$  $\Big)^2 \leq 2 \Big(\#$  $\Big) \cdot \Big(\#$  $\Big) \cdot \Big(\#$  $\Big)$.

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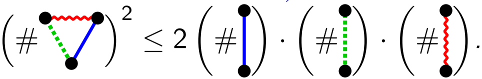
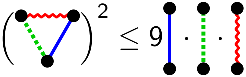
In 3-edge colored graph 

In flag algebras



Automated sum-of-squares proof needs 540GB RAM

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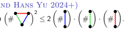
In 3-edge colored graph 

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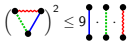
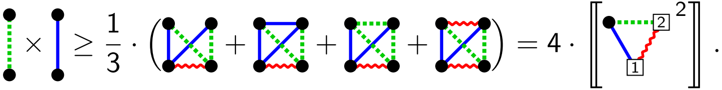


Automated sum-of-squares proof needs 540GB RAM

Motivation is for joins - one take lined colored reg, green and blue and asks for rainbow lines. Their proof uses entropy method.

1,601,952 configurations and 540GB ram

Balogh, Bradshaw, Garcia, L. 2025+



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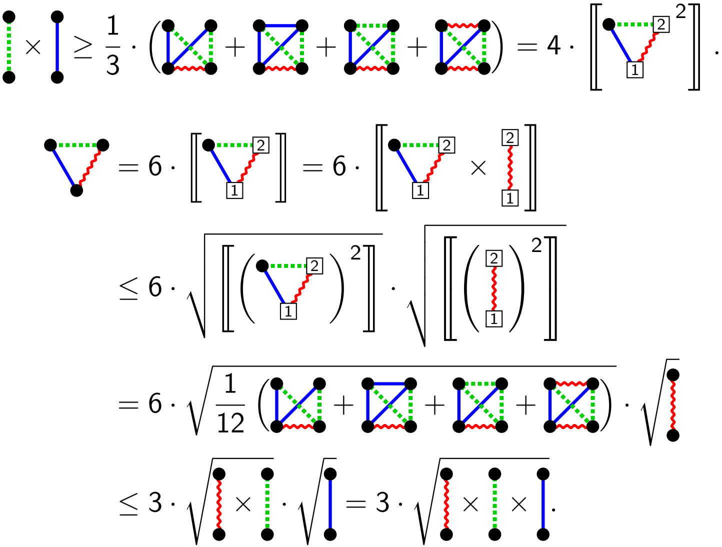
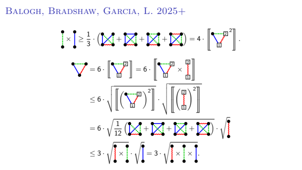
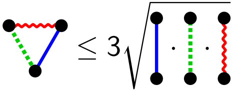
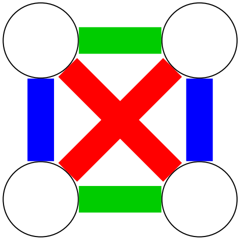




Can be written without flags, a simple counting proof.

Theorem (Chao and Hans Yu 2024+)





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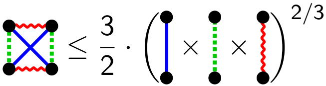
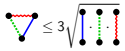


We also have exactness and translation to counting and a short entropy proofs.

Theorem (Chao and Hans Yu 2024+)



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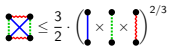
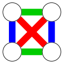
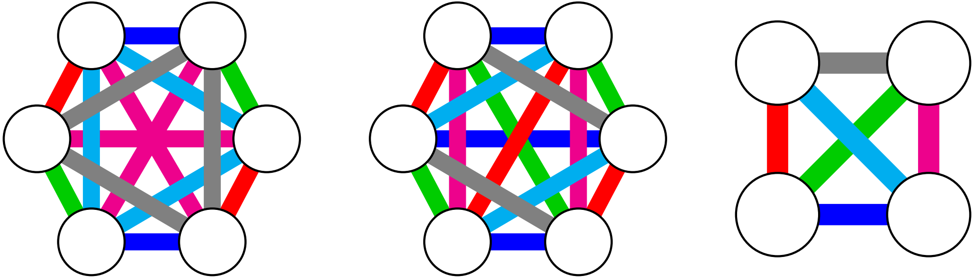




We also have exactness and translation to counting and a short entropy proofs.

We also have exactness results.

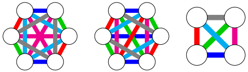
Further Directions



Question

Let G be a graph with edges colored by colors $\{1, \ldots, 6\}$. Denote by C_i the number of edges colored by color i. Let H be the number of rainbow copies of K_4 in G. Is it true that $H \le \sqrt[3]{\prod_i C_i}$?

Further Directions



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We note how to do it for a fixed rainbow coloring

Mathematical Biology

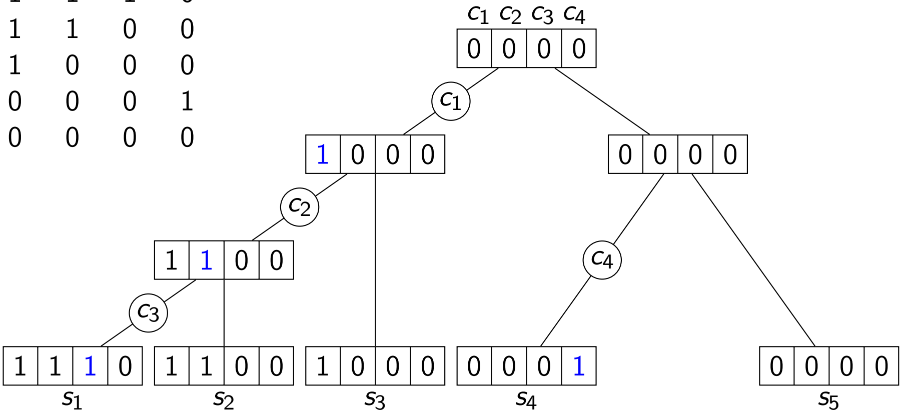
Phylogeny

How do we reconstruct an evolutionary history (*phylogeny*) from observations of living species from characters?

	Spine	Fur	Fins	Wings
Seal	1	1	1	0
Dog	1	1	0	0
Lizard	1	0	0	0
Butterfly	0	0	0	1
Worm	0	0	0	0

Phylogeny

	c_1	c_2	c_3	c_4
s_1	1	1	1	0
s_2	1	1	0	0
s_3	1	0	0	0
s_4	0	0	0	1
s_5	0	0	0	0



Assumptions and Problems

Perfect phylogeny model is fundamental, but inaccurate

Lemma (Three-Gamete Condition, Hudson-Kaplan '85)

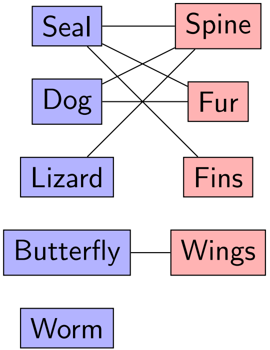
A set of species and characters has a perfect phylogeny if and only if for every pair of traits, no three species present all of the combinations $10, 01, 11$.

	Spine	Fins
Seal	1	1
Lizard	1	0
Butterfly	0	0

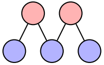
	Spine	Wings
Bird	1	1
Lizard	1	0
Butterfly	0	1

Incidence Graph

	Spine	Fur	Fins	Wings
Seal	1	1	1	0
Dog	1	1	0	0
Lizard	1	0	0	0
Butterfly	0	0	0	1
Worm	0	0	0	0

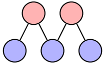


Claim

The incidence graph contains no induced copies of  *if and only if the three-gamete condition is satisfied*

Main Problem

Problem

How many induced copies of $M =$  *can we possibly have?*

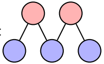
Counting M measures how far from perfect.

- Inducibility problem in *red-blue graphs*
- Bipartite graphs with fixed two-colorings
- Isomorphisms are graph isomorphisms and preserve colors

Results

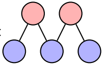
Theorem (Eulenstein, Halfpap, L., Miyasaki, Pfender, Volec 2025+)

Fix $\alpha > 0$. Let $G_{r,b}$ be a red-blue graph with r red vertices and b blue vertices with $\frac{r}{b} = \alpha$. Then

$\#$  $\leq \dfrac{r^2 b^3}{81} + o(r^2 b^3)$

Corollary (EHLMPV)

If G_n is a red-blue graph on n vertices then

$\#$  $\leq \dfrac{2^2 3^4 n^5}{15^5} + o(n^5).$

Asymptotic Extremal Examples

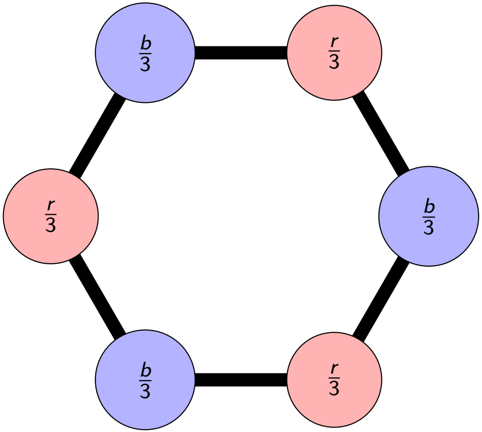


Figure: $C_6(r, b)$

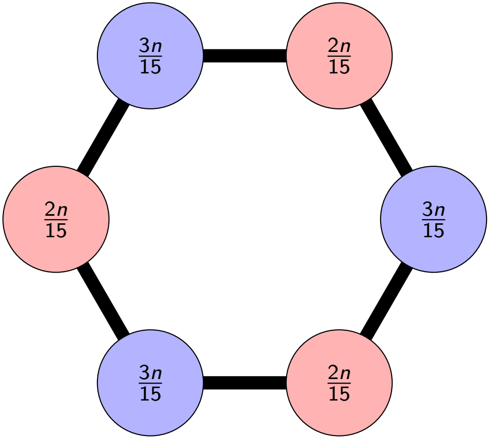
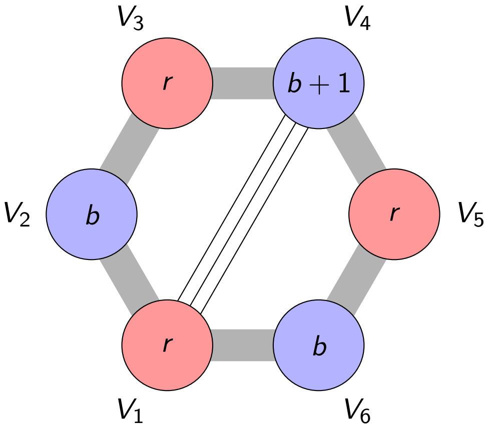


Figure: $C_6(n)$

Future Directions

- Determine lower-order terms and stronger characterizations



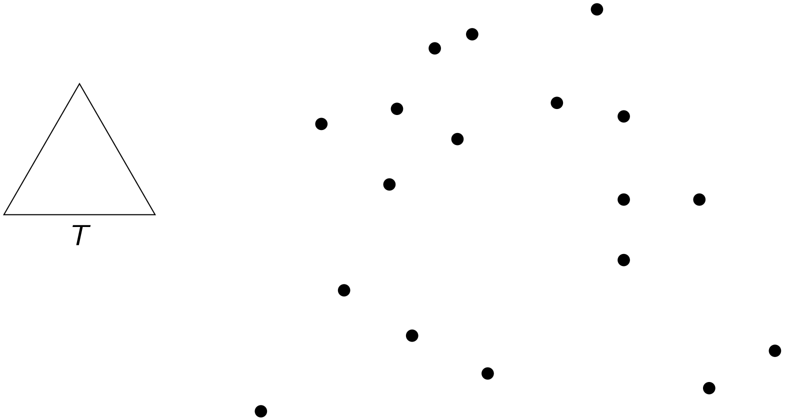
ε-similar Triangles

ε-similar Triangles

https://arxiv.org/abs/2101.10304

Problem

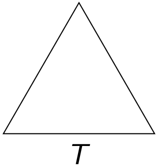
Let T be a triangle and $n \in \mathbb{N}$ fixed.
Which n points in $\mathbb{R}^2$ maximize the number of triangles similar to T?

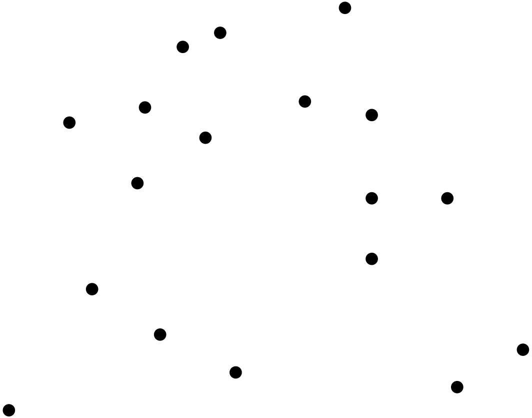


T_1 and T_2 are *ε-similar* if their inner angles differ by at most ε.
(OK to move, scale, rotate, ε-perturb)

Problem

Let T be a triangle and $n \in \mathbb{N}$ fixed. (and $\varepsilon > 0$ fixed)
Which n points in $\mathbb{R}^2$ maximize the number of triangles similar to T?





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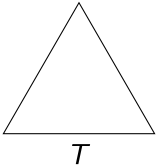

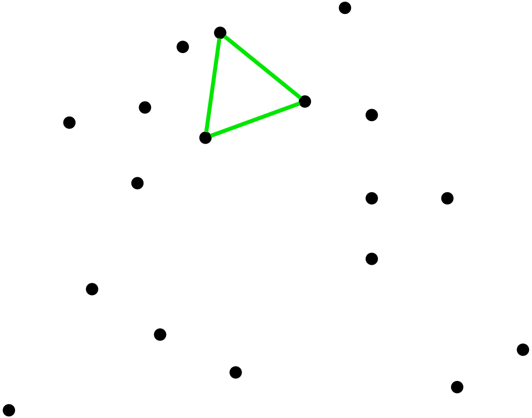

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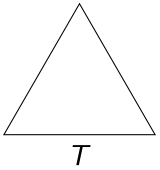

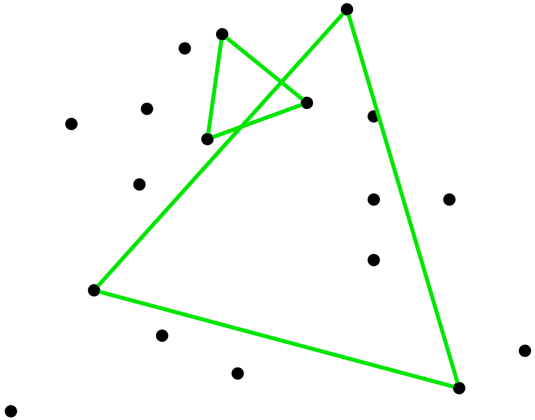

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(OK to move, scale, rotate, ε-perturb)

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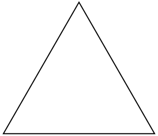
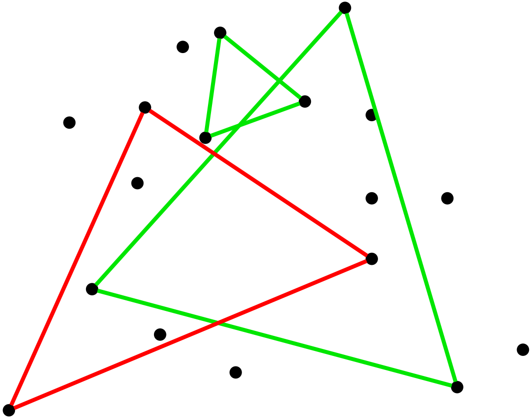


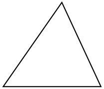
T



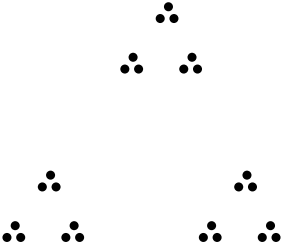
T_1 and T_2 are *ε-similar* if their inner angles differ by at most ε.
(OK to move, scale, rotate, ε-perturb)

Lower bound construction

Let T be a triangle and $n \in \mathbb{N}$ fixed. (and $\varepsilon > 0$ fixed)
Which n points in $\mathbb{R}^2$ maximize the number of triangles similar to T?



T



$h(n, T, \varepsilon) :=$ max # of *ε-similar* triangles to T, it is at least $\frac{1}{4}\binom{n}{3}(1 + o(1))$.

Lower bound construction

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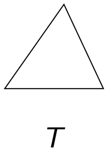
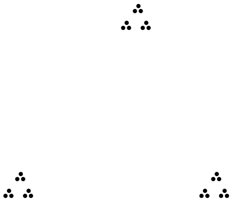




$h(n, T, \varepsilon) :=$ max # of *ε-similar* triangles to T, it is at least $\frac{1}{4}\binom{n}{3}(1 + o(1))$.

Results

Theorem (Bárány and Füredi (2019))

For almost every triangle T there is an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$,

$$h(n, T, \varepsilon) \leq 0.25072\binom{n}{3}(1 + o(1)).$$

If T is equilateral, then $h(n, T, \varepsilon) = \frac{1}{4}\binom{n}{3}(1 + o(1))$

Theorem (Balogh, Clemen, L. (2022))

For almost every triangle T there is an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$,

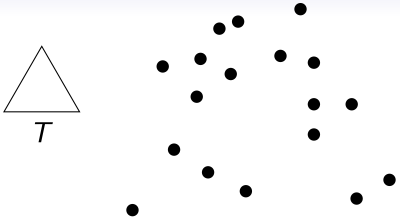
$$h(n, T, \varepsilon) = \frac{1}{4}\binom{n}{3}(1 + o(1)).$$

$h(n, T, \varepsilon) :=$ max # of *ε-similar* triangles to T, it is at least $\frac{1}{4}\binom{n}{3}(1 + o(1))$.

Let T and ε are given

- Fix n points in the plane.
- For every T' ε-similar to T, add a 3-edge
- Investigate the resulting hypergraph H

H has no subhypergaph in $\mathcal{F} = \{K_4^3, \ldots\}$



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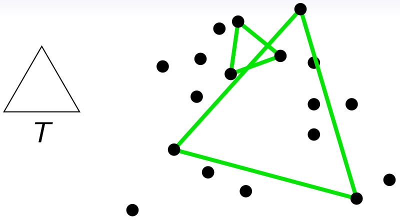

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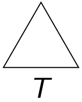

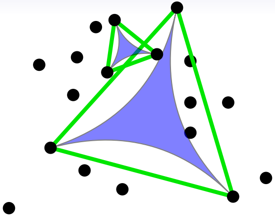

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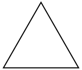
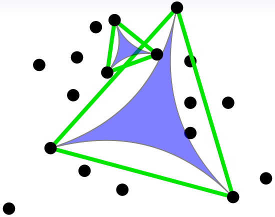


T



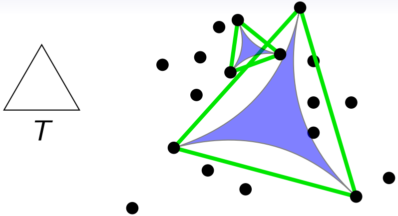
Theorem (Balogh, Clemen, L. (2022))

$\mathcal{F}$-free hypergraph has at most $\frac{1}{4}\binom{n}{3}(1+o(1))$ edges.

Let T and ε are given

- Fix n points in the plane.
- For every T' ε-similar to T, add a 3-edge
- Investigate the resulting hypergraph H

H has no subhypergaph in $\mathcal{F} = \{K_4^3, \dots\}$



Theorem (Balogh, Clemen, L. (2022))

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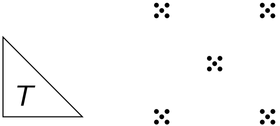
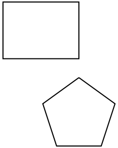
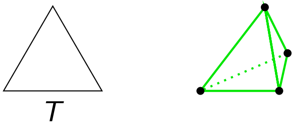
All triangles?



Other Shapes?



in $\mathbb{R}^d$?



Counting k-SAT functions

Question

Count functions

$$f : \{0,1\}^n \to \{0,1\} \qquad 2^{2^n}$$

k-SAT FUNCTION can be defined as

$$f(x_1, \dots, x_n) = C_1 \vee C_2 \vee \cdots \vee C_m$$

$$C_i = \underbrace{z_1 \wedge z_2 \wedge \cdots \wedge z_k}_{\text{all different variables}} \qquad z_i \in \{x_1, \neg x_1, x_2, \neg x_2, \dots, x_n, \neg x_n\}$$

x_i variable, C_i clause, z_i literal

example $k = 3$

$$x_1 \wedge x_2 \to (x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge \neg x_3)$$

$$x_1 \wedge x_2 \wedge \neg x_2 \to \text{always false}$$

Every k-SAT function has a formula but the formula may not be unique.

number of $f : \{0,1\}^n \to \{0,1\}$ 2^{2^n}

number of k-SAT formula $2^{2^k \binom{n}{k}}$

number of k-SAT functions?

k-SAT formula is *monotone* if it uses only $x_1, x_2, \ldots, x_n$, (i.e. no $\neg x_i$ is used)

All monotone k-SAT formula give different functions

$$g \not\ni x_1 \wedge \cdots \wedge x_k \ni f \qquad f \neq g \text{ at } x_1 = \cdots = x_k = 1, x_{k+1} = \cdots = x_m = 0$$

Number of monotone k-SAT functions $2^{\binom{n}{k}}$

k-SAT formula is *unate* if it uses at most one of $\{x_i, \neg x_i\} = \{x_i, \overline{x_i}\}$

Number of unate k-SAT functions $(1 + o(1))2^{n+\binom{n}{k}}$

Functions avoiding x_i counted multiple times

Conjecture (Bollobás, Brightwell, Leader 2003)

Fix $k \geq 2$, $1 - o(1)$ fraction of k-SAT functions are unate as $n \to \infty$. $(1 + o(1))2^{n+\binom{n}{k}}$

- # 2-SAT functions is $2^{(1+o(1))\binom{n}{2}}$. Bollobás, Brightwell, Leader 2003
  using Szemerédi regularity lemma
- Conjecture true for $k = 2$ Allen 2007
  using Szemerédi regularity lemma
- Conjecture true for $k = 2$ Ilinca, Kahn 2009
  without Szemerédi regularity lemma
- Conjecture true for $k = 3$ Ilinca, Kahn 2012
  using hypergraph regularity lemma
- Conjecture true for $k = 4, 5$ Dong, Mani, Zhao 2022

Conjecture true for all k :-) Balogh, Dong, Lidický, Mani, Zhao

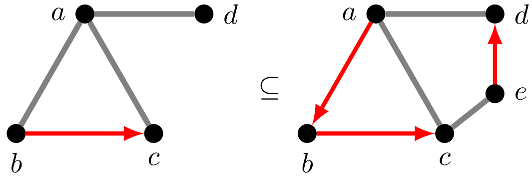
- Step 1: Reduction to a Turán type problem
  Dong, Mani, Zhao using blow-up, saturation, container method

- Step 2: Solving the extremal problem
  Balogh, Dong, L,, Many, Zhao: computer free flag-algebra

Partially directed k-graph is a k-uniform hypergraph, where every edge is

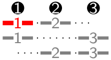
- undirected
- rooted at one vertex (directed towards one vertex)

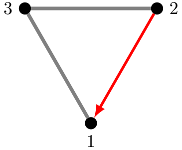
$\vec{H} \subseteq \vec{G}$ if $\vec{H}$ could be obtained from $\vec{G}$ by removing some vertices, edges, or orientations.



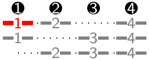
$\vec{T}_k$

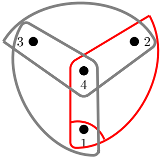
- $\vec{T}_2 = \{\hat{1}2, 13, 23\}$



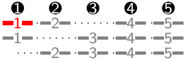


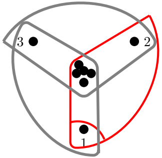
- $\vec{T}_3 = \{\hat{1}24, 134, 234\}$





- $\vec{T}_k = \{\hat{1}24 \cdots k+1, 134 \cdots k+1, 234 \cdots k+1\}$





Extremal problem

G is k-uniform, n-vertex, $\vec{T}_k$-free.

$$\alpha := \frac{e_{undirected}(G)}{\binom{n}{k}}$$

$$\beta := \frac{e_{directed}(G)}{\binom{n}{k}}$$

Given k, θ, what is

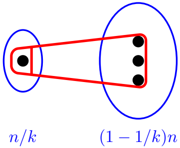
$$\max\{\alpha + \theta\beta\}?$$

Special (open) case:

Show $\alpha + \theta\beta \leq 1$ when $1 \leq \theta \leq \left(1 - \frac{1}{k}\right)^{1-k} \approx e$

Constructions:

Complete undirected graph



Conjecture (Bollobás, Brightwell, Leader 2003)

Fix $k \geq 2$, almost all k-SAT functions are unate.

Theorem (Dong, Mani, Zhao)

If $\alpha + \theta\beta \leq 1$ for some $\theta > \log_2 3$ then almost all k-SAT functions are unate.

This theorem is a lot of work. Uses hypergraph containers by Balogh, Morris, Samotij; Saxton, Thomasson (AMS Steele prize 2024)

Theorem (Dong, Mani, Zhao)

Conjecture true for $k \leq 5$.

Theorem (Balogh, Dong, Lidický, Mani, Zhao)

Conjecture true for all k.

Proof for $k \geq 4$ using flag algebras

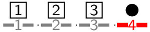
If graphs represent densities as

 $= \alpha := \dfrac{e_{undirected}(G)}{\binom{n}{k}}$  $= \beta := \dfrac{e_{directed}(G)}{\binom{n}{k}}$

then

$\alpha + \theta\beta$

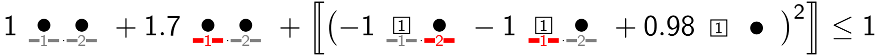
$=$  $+ \theta$ 

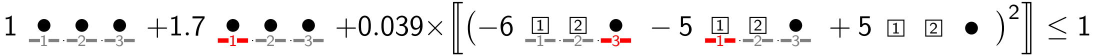
$\leq$  $+ \theta$  $+ \Big[\!\Big[\Big(a$  $- b$  $\Big)^2 \Big]\!\Big]$

≤ 1

for $\theta = 1 + \frac{1}{\sqrt{2}} \geq 1.707 > \log_2 3 \quad a = \frac{1}{\sqrt{2}}, b = \frac{k(\theta-1)-1}{\sqrt{2}} \quad k \geq 4$

Proof for $k = 2$ and $k = 3$





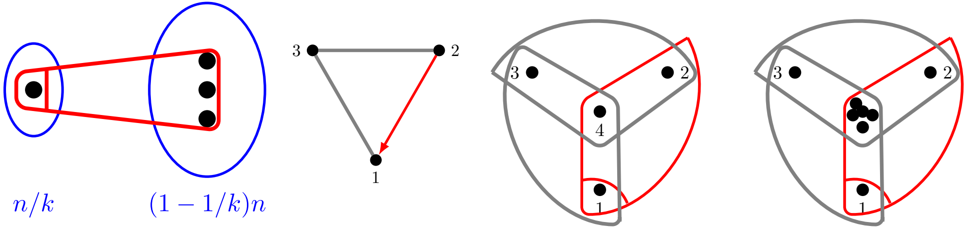
Future Directions

Theorem (Balogh, Dong, Lidický, Mani, Zhao)

If $\vec{T}_k$ is forbidden, then  $+\left(1+\frac{1}{\sqrt{2}}\right)$  ≤ 1 *for all k.*

Question

If $\vec{T}_k$ is forbidden, then  $+\left(1-\frac{1}{k}\right)^{1-k}$  ≤ 1 *for all k?*



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If you rerun the document (without altering it) this surplus page will go away, becau LaTeX now knows how many pages to expect for this document.