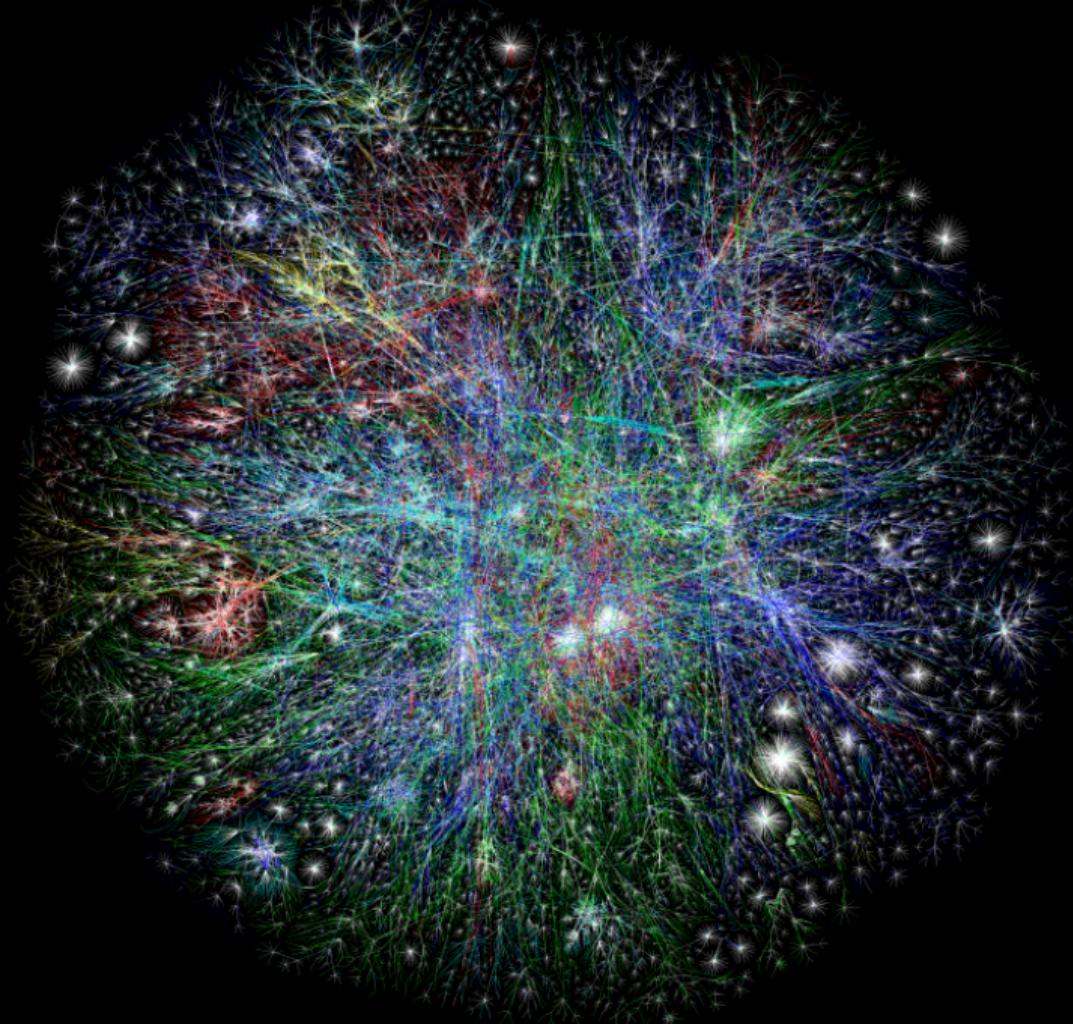


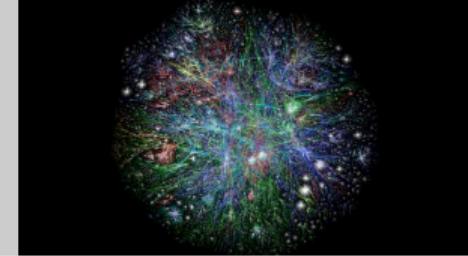
FLAG ALGEBRAS AND ITS APPLICATION

Bernard Lidický



International Conference in Discrete Mathematics
Tiruchirappalli, India
Dec 11, 2025



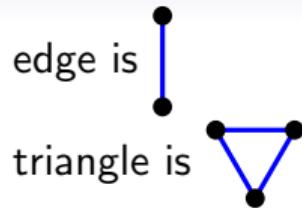


We will only consider large graphs (or networks).

Here is a graph of the internet from a while back to show there are large graphs that are interesting.

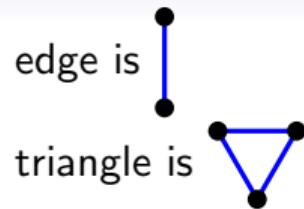
INSPIRATIONAL PROBLEM

- Let n be a fixed number of vertices in a graph G .
- Assume G has m edges.
- What is the number of triangles in G ?



INSPIRATIONAL PROBLEM

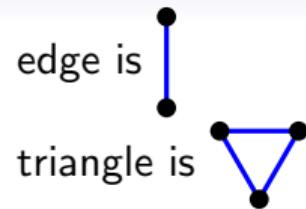
- Let n be a fixed number of vertices in a graph G .
- Assume G has m edges. $\in [0, \binom{n}{2}]$
- What is the number of triangles in G ? $\in [0, \binom{n}{3}]$



INSPIRATIONAL PROBLEM

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Liu, Pikhurko, Staden 2020 (144 pages)



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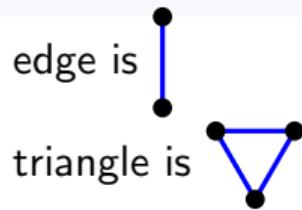
Liu, Pikhurko, Staden 2020 (144 pages)

Consider $n \rightarrow \infty$.

$$\# \text{ Edges} = p \binom{n}{2}$$

$$\# \text{ Triangles} = t \binom{n}{3}$$

Now $p, t \in [0, 1]$.



INSPIRATIONAL PROBLEM

- Let n be a fixed number of vertices in a graph G .
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Liu, Pikhurko, Staden 2020 (144 pages)

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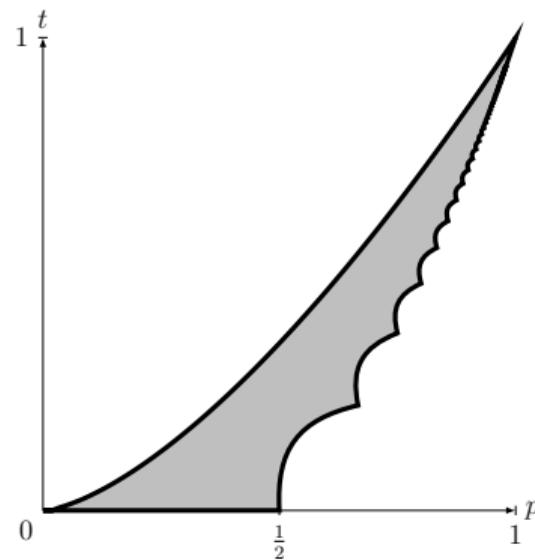
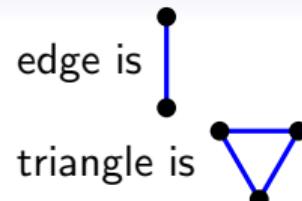
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Upper bound $p^{3/2}$ Kruskal-Katona 1964

Asymptotic lower bound by Razborov 2008



└ Inspirational Problem

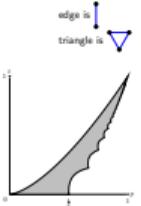
The edge $(0,0)$ to $(1/2,0)$ is Mantel's theorem. The sharp points are Turán graphs. The figure is exaggerated to show the idea if the shape.

INSPIRATIONAL PROBLEM

- Let n be a fixed number of vertices in a graph G .
- Assume G has m edges $\in [0, \binom{n}{2}]$
- What is the number of triangles in G ? $\in [0, \binom{m}{3}]$

Lia, Pilhurko, Staden 2020 (144 pages)
Consider $n \rightarrow \infty$.
Edges $\approx p \binom{n}{2}$
Triangles $\approx t \binom{p}{3}$
Now $p, t \in [0, 1]$.

Upper bound $p^{3/2}$ Kruskal-Katona 1964
Asymptotic lower bound by Razborov 2008



FLAG ALGEBRAS

Seminal paper:

Razborov, Flag Algebras, *Journal of Symbolic Logic* 72 (2007), 1239–1282.

David P. Robbins Prize by AMS for Razborov in 2013 over 300 citations (on Google)



EXAMPLE

If density of edges is p , what is the minimum density of triangles?

- Designed to attack extremal problems.
- Works well if constraints as well as desired value can be computed by checking small subgraphs (or average over small subgraphs).
- The results are for the limit as graphs get very large.

EXAMPLE EXTREMAL PROBLEM



THEOREM (MANTEL 1907)

Every n -vertex triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

PROBLEM

Maximize a graph parameter (# of edges) over a class of graphs (triangle-free).

- local condition and global parameter (computable locally)
- threshold
- bound and extremal example

PROOF OF MANTEL'S THEOREM

THEOREM (MANTEL 1907)

In every n -vertex triangle-free graph $|E| \leq \frac{1}{4}n^2$.

PROOF.

$$n|E| \geq \sum_{ij \in E} \underbrace{(d_i + d_j)}_{\leq n} = \sum_{i \in V} d_i^2 \geq \frac{(\sum_{i \in V} d_i)^2}{n} = \frac{4|E|^2}{n}$$

□

Cauchy-Schwarz $(\sum_i a_i b_i)^2 \leq \sum_i a_i^2 \cdot \sum_i b_i^2$ with $b_i = 1$.

Cauchy-Schwarz $(\sum_i a_i 1)^2 \leq \sum_i a_i^2 \cdot \sum_i 1^2$.

Flag Algebras and Its Application

└ Flag Algebras

└ Proof of Mantel's Theorem

$|E|$ is number of edges

d_i is a degree of a vertex i . $d_i + d_j \leq n$ because the graph is triangle-free.

We will try to rewrite the proof using densities and this should get us familiar with flag algebras notation.

PROOF OF MANTEL'S THEOREM

THEOREM (MANTEL 1907)

In every n -vertex triangle-free graph $|E| \leq \frac{1}{4}n^2$.

PROOF.

$$n|E| \geq \sum_{\substack{i \in V \\ j \in V \\ i \neq j}} (d_i + d_j) = \sum_{i \in V} d_i^2 \geq \frac{(\sum_{i \in V} d_i)^2}{n} = \frac{4|E|^2}{n}$$

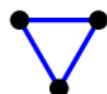
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Cauchy-Schwarz $(\sum_{i \in V} a_i 1)^2 \leq \sum_{i \in V} a_i^2 \cdot \sum_{i \in V} 1^2$.

□

FLAG ALGEBRAS DEFINITIONS

Let G be a graph on n vertices.



The probability that three random vertices in G span a triangle, i.e.
 $\#\triangle / \binom{n}{3}$.

Flag Algebras and Its Application

└ Flag Algebras

└ Flag algebras definitions

The last click the = 1 is for audience participation.

FLAG ALGEBRAS DEFINITIONS

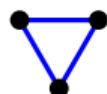
Let G be a graph on n vertices.



The probability that three random vertices in G span a triangle, i.e.
 $\mathbb{P}(\triangle) = \mathbb{P}(\binom{3}{2})$.

FLAG ALGEBRAS DEFINITIONS

Let G be a graph on n vertices.



The probability that three random vertices in G span a triangle, i.e.
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The probability that three random vertices in G span exactly two edges.
 $\# \text{triangle with one edge missing} / \binom{n}{3}$.

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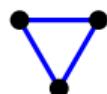
The probability that three random vertices in G span a triangle, i.e.
 $\# \text{Flag} / \binom{n}{3}$.



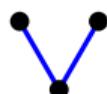
The probability that three random vertices in G span exactly two
edges. $\# \text{Flag} / \binom{n}{3}$.

FLAG ALGEBRAS DEFINITIONS

Let G be a graph on n vertices.



The probability that three random vertices in G span a triangle, i.e.
 $\# \triangle / \binom{n}{3}$.



The probability that three random vertices in G span exactly two edges. $\# \text{V-shape} / \binom{n}{3}$.



The probability that a random vertex other than 1 is adjacent to 1
 $= \deg(1) / (n - 1)$.

Flag Algebras and Its Application

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FLAG ALGEBRAS DEFINITIONS

Let G be a graph on n vertices.



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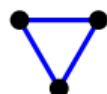
The probability that three random vertices in G span exactly two
edges. $\# \nabla / \binom{n}{3}$.



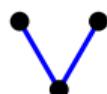
The probability that a random vertex other than \square is adjacent to \square
 $= \text{deg}(\square) / (n - 1)$.

FLAG ALGEBRAS DEFINITIONS

Let G be a graph on n vertices.



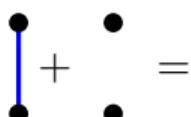
The probability that three random vertices in G span a triangle, i.e.
 $\# \triangle / \binom{n}{3}$.



The probability that three random vertices in G span exactly two edges. $\# \text{triangle} / \binom{n}{3}$.



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Flag Algebras and Its Application

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FLAG ALGEBRAS DEFINITIONS

Let G be a graph on n vertices.



The probability that three random vertices in G span a triangle, i.e.
 $\#\text{ } \triangle / \binom{n}{3}$.



The probability that three random vertices in G span exactly two
edges. $\#\text{ } \text{V} / \binom{n}{3}$.

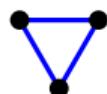


The probability that a random vertex other than \square is adjacent to \square
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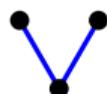


FLAG ALGEBRAS DEFINITIONS

Let G be a graph on n vertices.



The probability that three random vertices in G span a triangle, i.e.
 $\# \triangle / \binom{n}{3}$.



The probability that three random vertices in G span exactly two edges. $\# \text{triangle} / \binom{n}{3}$.



The probability that a random vertex other than 1 is adjacent to 1
 $= \deg(1) / (n - 1)$.

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \end{array} = 1$$

Flag Algebras and Its Application

└ Flag Algebras

└ Flag algebras definitions

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FLAG ALGEBRAS DEFINITIONS

Let G be a graph on n vertices.



The probability that three random vertices in G span a triangle, i.e.
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The probability that three random vertices in G span exactly two
edges. $\# \text{ } \text{V} / \binom{n}{3}$.

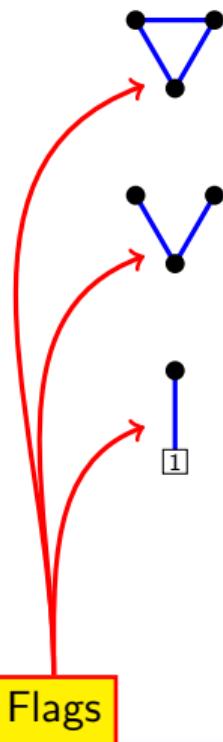


The probability that a random vertex other than \square is adjacent to \square
 $= \text{deg}(\square) / (n - 1)$.

$$\begin{bmatrix} + & : & : \end{bmatrix} = 1$$

FLAG ALGEBRAS DEFINITIONS

Let G be a graph on n vertices.



The probability that three random vertices in G span a triangle, i.e.
 $\#\triangle / \binom{n}{3}$.

The probability that three random vertices in G span exactly two edges. $\# \text{path of 3} / \binom{n}{3}$.

The probability that a random vertex other than 1 is adjacent to 1
 $= \deg(1) / (n - 1)$.

$$\begin{array}{c} \bullet \\ \bullet \\ \end{array} + \begin{array}{c} \bullet \\ \bullet \\ \end{array} = 1$$

Flag Algebras and Its Application

└ Flag Algebras

└ Flag algebras definitions

FLAG ALGEBRAS DEFINITIONS

Let G be a graph on n vertices.



The probability that three random vertices in G span a triangle, i.e.
 $= \frac{|\{ \text{triangle} \}|}{|\{ \text{3 vertices} \}|}$.

The probability that three random vertices in G span exactly two edges. $= \frac{|\{ \text{2 edges} \}|}{|\{ \text{3 vertices} \}|}$.

The probability that a random vertex other than \square is adjacent to \square
 $= \text{deg}(\square)/(n-1)$.

$$\begin{bmatrix} + & : \\ : & : \end{bmatrix} = 1$$

Flags

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THEOREM (MANTEL 1907)

In every n -vertex triangle-free graph $|E| \leq \frac{1}{4}n^2$.

$$n^2|E| \geq n \sum_{ij \in E} \underbrace{(d_i + d_j)}_{\leq n} = \sum_{i \in V} 1^2 \cdot \sum_{i \in V} d_i^2 \geq \left(\sum_{i \in V} 1 \cdot d_i\right)^2 = 4|E|^2$$

$$|E| = \binom{n}{2} \approx \frac{n^2}{2}, \quad d_1 = (n-1) \approx \frac{1}{1} \cdot n, \quad 1 = \frac{1}{1}$$

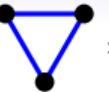
THEOREM (MANTEL 1907)

In every n -vertex triangle-free graph $|E| \leq \frac{1}{4}n^2$. If  = 0 then  $\leq \frac{1}{2}$

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$$n^2 \times \left| \frac{n^2}{2} \right| \geq \sum_{\boxed{1} \in V} \boxed{1}^2 \cdot \sum_{\boxed{1} \in V} \left(\frac{\boxed{1}}{\boxed{1}} n \right)^2 \geq \left(\sum_{\boxed{1} \in V} \boxed{1} \cdot \frac{\boxed{1}}{\boxed{1}} n \right)^2 = 4 \left(\frac{n^2}{2} \right)^2$$

$$|E| = \left| \frac{\binom{n}{2}}{2} \right| \approx \left| \frac{\frac{n^2}{2}}{2} \right|, \quad d_1 = \left| \frac{\boxed{1}}{1} (n-1) \right| \approx \left| \frac{\boxed{1}}{\boxed{1}} \cdot n \right|, \quad 1 = \boxed{1}$$

THEOREM (MANTEL 1907)

In every n -vertex triangle-free graph $|E| \leq \frac{1}{4}n^2$. If  = 0 then  $\leq \frac{1}{2}$

$$n^2|E| \geq n \sum_{ij \in E} \underbrace{(d_i + d_j)}_{\leq n} = \sum_{i \in V} 1^2 \cdot \sum_{i \in V} d_i^2 \geq \left(\sum_{i \in V} 1 \cdot d_i \right)^2 = 4|E|^2$$

$$n^2 \times \left| \frac{n^2}{2} \right| \geq \sum_{\square \in V} \square^2 \cdot \sum_{\square \in V} \left(\frac{n}{\square} \right)^2 \geq \left(\sum_{\square \in V} \square \cdot \frac{n}{\square} \right)^2 = 4 \left(\frac{n^2}{2} \right)^2$$

$$\frac{1}{2} \left| \frac{n^2}{2} \right| \geq \frac{1}{n} \sum_{\square \in V} \square^2 \cdot \frac{1}{n} \sum_{\square \in V} \left(\frac{n}{\square} \right)^2 \geq \left(\frac{1}{n} \sum_{\square \in V} \square \cdot \frac{n}{\square} \right)^2 = \left| \frac{n^2}{2} \right|^2$$

$$|E| = \left| \frac{n^2}{2} \right| \approx \frac{n^2}{2}, \quad d_1 = \frac{n^2}{2} (n-1) \approx \frac{n^2}{2} \cdot n, \quad 1 = \square$$

Flag Algebras and Its Application

└ Flag Algebras

THEOREM (MANTEL 1907)

In every n -vertex triangle-free graph $|E| \leq \frac{1}{4}n^2$. If $\nabla \triangle = 0$ then $\square \leq \frac{1}{2}$

$$n^2|E| \geq n \sum_{\substack{d_i, d_j \\ \leq 2}} (d_i + d_j) = \sum_{i \in V} 1^2 \cdot \sum_{j \in V} d_j^2 \geq (\sum_{i \in V} 1 \cdot d_i)^2 = 4|E|^2$$

$$n^2 \times \frac{n^2}{2} \geq \sum_{\square \in V} \square^2 \cdot \sum_{\square \in V} \left(\frac{\square}{\square} \right)^2 \geq \left(\sum_{\square \in V} \square \cdot \frac{\square}{\square} \right)^2 = 4 \left(\frac{n^2}{2} \right)^2$$

$$\frac{1}{2} \square \geq \frac{1}{n} \sum_{\square \in V} \square^2 \cdot \frac{1}{n} \sum_{\square \in V} \left(\frac{\square}{\square} \right)^2 \geq \left(\frac{1}{n} \sum_{\square \in V} \square \cdot \frac{\square}{\square} \right)^2 = \square^2$$

$$|E| = \binom{n}{2} \approx \frac{n^2}{2}, \quad d_1 = \frac{n}{2}(n-1) \approx \frac{n^2}{2} \cdot n, \quad 1 = \square$$

We are ignoring lower order terms and approximate $\binom{n}{k}$ by $\frac{n^k}{k!}$.

$$n^2|E| \geq n \sum_{ij \in E} \underbrace{(d_i + d_j)}_{\leq n} = \sum_{i \in V} 1^2 \cdot \sum_{i \in V} d_i^2 \geq (\sum_{i \in V} 1 \cdot d_i)^2 = 4|E|^2$$

$$\frac{1}{2} \begin{array}{c} \bullet \\ \textcolor{blue}{|} \\ \bullet \end{array} \geq \frac{1}{n} \sum_{\boxed{1} \in V} \boxed{1}^2 \cdot \frac{1}{n} \sum_{\boxed{1} \in V} \left(\begin{array}{c} \bullet \\ \textcolor{blue}{|} \\ \boxed{1} \end{array} \right)^2 \geq \left(\frac{1}{n} \sum_{\boxed{1} \in V} \boxed{1} \cdot \begin{array}{c} \bullet \\ \textcolor{blue}{|} \\ \boxed{1} \end{array} \right)^2 = \begin{array}{c} \bullet \\ \textcolor{blue}{|} \\ \bullet \end{array}^2$$

$$\frac{1}{n} \sum_{\boxed{1} \in V} f = \llbracket f \rrbracket$$

Cauchy-Schwarz: $\llbracket f^2 \rrbracket \cdot \llbracket g^2 \rrbracket \geq \llbracket f \cdot g \rrbracket^2$. In particular, $\llbracket f^2 \rrbracket \geq 0$.

$$n^2|E| \geq n \sum_{ij \in E} \underbrace{(d_i + d_j)}_{\leq n} = \sum_{i \in V} 1^2 \cdot \sum_{i \in V} d_i^2 \geq (\sum_{i \in V} 1 \cdot d_i)^2 = 4|E|^2$$

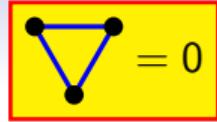
$$\frac{1}{2} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \geq \frac{1}{n} \sum_{\boxed{1} \in V} \boxed{1}^2 \cdot \frac{1}{n} \sum_{\boxed{1} \in V} \left(\begin{array}{c} \bullet \\ \boxed{1} \end{array} \right)^2 \geq \left(\frac{1}{n} \sum_{\boxed{1} \in V} \boxed{1} \cdot \begin{array}{c} \bullet \\ \boxed{1} \end{array} \right)^2 = \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array}^2$$

$$\frac{1}{2} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \geq \llbracket \boxed{1}^2 \rrbracket \cdot \llbracket \begin{array}{c} \bullet \\ \boxed{1} \end{array}^2 \rrbracket \geq \llbracket \boxed{1} \cdot \begin{array}{c} \bullet \\ \boxed{1} \end{array} \rrbracket^2 = \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array}^2$$

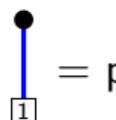
$$\frac{1}{n} \sum_{\boxed{1} \in V} f = \llbracket f \rrbracket$$

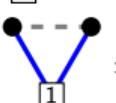
Cauchy-Schwarz: $\llbracket f^2 \rrbracket \cdot \llbracket g^2 \rrbracket \geq \llbracket f \cdot g \rrbracket^2$. In particular, $\llbracket f^2 \rrbracket \geq 0$.

$$\frac{1}{2} \geq \left[\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix} \right] \cdot \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right]$$

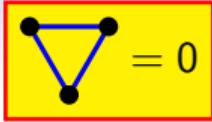


$\boxed{1}$ = not choosing anything = 1

 = probability of choosing a vertex ... $\deg(\boxed{1})/(n-1)$

 = probability of choosing two distinct vertices ... $\binom{\deg(\boxed{1})}{2} / \binom{n-1}{2}$

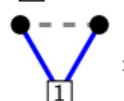
$$\frac{1}{2} \geq \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right]^2 \cdot \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \right]$$



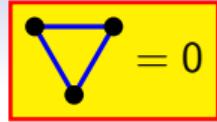
$$\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right] \times \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right] = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right] + o(1) = \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \right] + \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right] + o(1)$$

$\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right]$ = not choosing anything = 1

 = probability of choosing a vertex ... $\deg(\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right])/(n-1)$

 = probability of choosing two distinct vertices ... $\binom{\deg(\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right])}{2} / \binom{n-1}{2}$

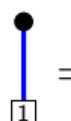
$$\frac{1}{2} \geq \left[\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right]^2 \right] \cdot \left[\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \right] \right] = 1 \cdot \left[\left[\begin{smallmatrix} \bullet & \bullet \\ 1 & 1 \end{smallmatrix} \right] + \left[\begin{smallmatrix} \bullet & \bullet \\ 1 & 1 \end{smallmatrix} \right] \right] = \left[\left[\begin{smallmatrix} \bullet & \bullet \\ 1 & 1 \end{smallmatrix} \right] \right]$$

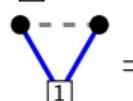


$$= 0$$

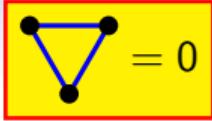
$$\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right] \times \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right] = \left[\begin{smallmatrix} \bullet & \bullet \\ 1 & 1 \end{smallmatrix} \right] + o(1) = \left[\begin{smallmatrix} \bullet & \bullet \\ 1 & 1 \end{smallmatrix} \right] + \left[\begin{smallmatrix} \bullet & \bullet \\ 1 & 1 \end{smallmatrix} \right] + o(1)$$

$\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$ = not choosing anything = 1

 = probability of choosing a vertex ... $\deg(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix})/(n-1)$

 = probability of choosing two distinct vertices ... $\binom{\deg(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix})}{2} / \binom{n-1}{2}$

$$\frac{1}{2} \geq \left[\begin{smallmatrix} \bullet \\ \square \end{smallmatrix} \right] \cdot \left[\begin{smallmatrix} \bullet & 2 \\ \square & 1 \end{smallmatrix} \right] = 1 \cdot \left[\begin{smallmatrix} \bullet & \bullet \\ \square & 1 \end{smallmatrix} + \begin{smallmatrix} \bullet & \bullet \\ \square & 1 \end{smallmatrix} \right] = \left[\begin{smallmatrix} \bullet & \bullet \\ \square & 1 \end{smallmatrix} \right]$$



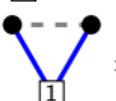
$$= 0$$

$$\left[\begin{smallmatrix} \bullet \\ \square \end{smallmatrix} \right] \times \left[\begin{smallmatrix} \bullet \\ \square \end{smallmatrix} \right] = \left[\begin{smallmatrix} \bullet & \bullet \\ \square & 1 \end{smallmatrix} \right] + o(1) = \left[\begin{smallmatrix} \bullet & \bullet \\ \square & 1 \end{smallmatrix} \right] + \left[\begin{smallmatrix} \bullet & \bullet \\ \square & 1 \end{smallmatrix} \right] + o(1)$$

$$\left[\begin{smallmatrix} \bullet \\ \square \end{smallmatrix} \right] \times \left[\begin{smallmatrix} \bullet \\ \square \end{smallmatrix} \right] = \frac{1}{2} \left[\begin{smallmatrix} \bullet & \bullet \\ \square & 1 \end{smallmatrix} \right] = \frac{1}{2} \left[\begin{smallmatrix} \bullet & \bullet \\ \square & 1 \end{smallmatrix} \right] + \frac{1}{2} \left[\begin{smallmatrix} \bullet & \bullet \\ \square & 1 \end{smallmatrix} \right]$$

\square = not choosing anything = 1

 = probability of choosing a vertex ... $\deg(\square)/n$

 = probability of choosing two distinct vertices ... $\binom{\deg(\square)}{2} / \binom{n-1}{2}$

Flag Algebras and Its Application

└ Flag Algebras

$$\frac{1}{2} \begin{smallmatrix} \bullet \\ \square \end{smallmatrix} \geq \left[\begin{smallmatrix} \bullet \\ \square \end{smallmatrix} \right]^2 \cdot \left[\begin{smallmatrix} \square \\ \square \end{smallmatrix} \right]^2 = 1 \cdot \left[\begin{smallmatrix} \square \\ \square \end{smallmatrix} \right] = \left[\begin{smallmatrix} \square \\ \square \end{smallmatrix} \right]$$

$$\begin{smallmatrix} \bullet \\ \square \end{smallmatrix} \times \begin{smallmatrix} \bullet \\ \square \end{smallmatrix} = \begin{smallmatrix} \square \\ \square \end{smallmatrix} + o(1) = \begin{smallmatrix} \bullet \\ \square \end{smallmatrix} + \begin{smallmatrix} \square \\ \bullet \end{smallmatrix} + o(1)$$

$$\begin{smallmatrix} \bullet \\ \square \end{smallmatrix} \times \begin{smallmatrix} \bullet \\ \bullet \end{smallmatrix} = \frac{1}{2} \begin{smallmatrix} \bullet \\ \square \end{smallmatrix} + \frac{1}{2} \begin{smallmatrix} \bullet \\ \bullet \end{smallmatrix} = \frac{1}{2} \begin{smallmatrix} \bullet \\ \square \end{smallmatrix} + \frac{1}{2} \begin{smallmatrix} \bullet \\ \bullet \end{smallmatrix}$$

\square = not choosing anything = 1

\bullet = probability of choosing a vertex ... $\text{deg}(\square)/(n-1)$

$\begin{smallmatrix} \bullet \\ \square \end{smallmatrix}$ = probability of choosing two distinct vertices ... $\frac{\text{deg}(\square)}{2} / \binom{n-1}{2}$

Probability of choosing a vertex inducing an edge with the fixed vertex 1.

Probability of choosing a pair of distinct vertices each being in an edge with the fixed vertex 1.

Notice on the left each pair of vertices counted twice!

We will ignore $o(1)$ in the future

$$\frac{1}{2} \geq \left[\begin{array}{c} \bullet \\ \bullet \\ \hline 1 \end{array} \right]$$

$$\sum_{\substack{\bullet \\ \square \in V}} \left[\begin{array}{c} \bullet \\ \bullet \\ \hline \square \end{array} \right] \binom{n-1}{2} = \# \left[\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \hline \end{array} \right]$$

$$\sum_{\substack{\bullet \\ \square \in V}} \left[\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \hline \square \end{array} \right] = \text{probability of choosing three distinct vertices} \dots \# \left[\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \hline \end{array} \right] / \binom{n}{3}$$

$$\frac{1}{2} \geq \left[\left[\begin{array}{c} \bullet \\ \square \\ \bullet \end{array} \right] \right]$$

$$\left[\left[\begin{array}{c} \bullet \\ \square \\ \bullet \end{array} \right] \right] = \frac{1}{n} \sum_{\square \in V} \left[\begin{array}{c} \bullet \\ \square \\ \bullet \end{array} \right] = \frac{1}{3} \left[\begin{array}{c} \bullet \\ \square \\ \bullet \end{array} \right]$$

$$\left[\begin{array}{c} \bullet \\ \square \\ \bullet \end{array} \right] = \text{probability of choosing two distinct vertices} \dots \binom{\deg(\square)}{2} / \binom{n-1}{2}$$

$$\sum_{\square \in V} \left[\begin{array}{c} \bullet \\ \square \\ \bullet \end{array} \right] \binom{n-1}{2} = \# \left[\begin{array}{c} \bullet \\ \square \\ \bullet \end{array} \right]$$

$$\left[\begin{array}{c} \bullet \\ \square \\ \bullet \end{array} \right] = \text{probability of choosing three distinct vertices} \dots \# \left[\begin{array}{c} \bullet \\ \square \\ \bullet \end{array} \right] / \binom{n}{3}$$

$$\frac{1}{2} \geq \left[\left[\begin{array}{c} \bullet \\ \square \\ \bullet \end{array} \right] \right] = \frac{1}{3}$$

$$\left[\left[\begin{array}{c} \bullet \\ \square \\ \bullet \end{array} \right] \right] = \frac{1}{n} \sum_{\square \in V} \left[\begin{array}{c} \bullet \\ \square \\ \bullet \end{array} \right] = \frac{1}{3}$$

$$\left[\begin{array}{c} \bullet \\ \square \\ \bullet \end{array} \right] = \text{probability of choosing two distinct vertices} \dots \binom{\deg(\square)}{2} / \binom{n-1}{2}$$

$$\sum_{\square \in V} \left[\begin{array}{c} \bullet \\ \square \\ \bullet \end{array} \right] \binom{n-1}{2} = \# \left[\begin{array}{c} \bullet \\ \square \\ \bullet \end{array} \right]$$

$$\left[\begin{array}{c} \bullet \\ \square \\ \bullet \end{array} \right] = \text{probability of choosing three distinct vertices} \dots \# \left[\begin{array}{c} \bullet \\ \square \\ \bullet \end{array} \right] / \binom{n}{3}$$

$$\frac{1}{2} \geq \left[\begin{array}{c} \bullet \\ \bullet \\ \hline 1 \end{array} \right] = \frac{1}{3}$$

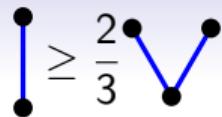
$$\geq \frac{2}{3}$$

$$\left[\begin{array}{c} \bullet \\ \bullet \\ \hline 1 \end{array} \right] = \frac{1}{n} \sum_{1 \in V} \left[\begin{array}{c} \bullet \\ \bullet \\ \hline 1 \end{array} \right] = \frac{1}{3}$$

$$\left[\begin{array}{c} \bullet \\ \bullet \\ \hline 1 \end{array} \right] = \text{probability of choosing two distinct vertices} \dots \binom{\deg(1)}{2} / \binom{n-1}{2}$$

$$\sum_{1 \in V} \left[\begin{array}{c} \bullet \\ \bullet \\ \hline 1 \end{array} \right] \binom{n-1}{2} = \# \left[\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \hline 1 \end{array} \right]$$

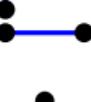
$$\left[\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \hline 1 \end{array} \right] = \text{probability of choosing three distinct vertices} \dots \# \left[\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \hline 1 \end{array} \right] / \binom{n}{3}$$

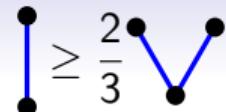


$$|E| = \frac{1}{n} \left(\sum_{\bullet \bullet} 1 + \sum_{\bullet \vee \bullet} 2 + \sum_{\bullet \triangle \bullet} 3 \right)$$

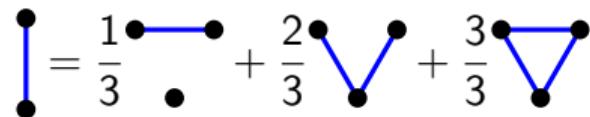
$$\binom{n}{2} \approx \frac{1}{n} \bullet \bullet \binom{n}{3} + \frac{2}{n} \bullet \vee \bullet \binom{n}{3} + \frac{3}{n} \bullet \triangle \bullet \binom{n}{3}$$

 = probability of choosing an edge ... $|E|/\binom{n}{2}$

 = probability of choosing an triple ... $\# \bullet \vee \bullet / \binom{n}{3}$



$$\geq \frac{2}{3}$$

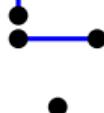


$$|E| = \frac{1}{n} \left(\sum_{\substack{\bullet \\ \vdots \\ \bullet}} 1 + \sum_{\substack{\bullet \\ \vee \\ \bullet}} 2 + \sum_{\substack{\bullet \\ \triangle \\ \bullet}} 3 \right)$$

$$\binom{n}{2} \approx \frac{1}{n} \bullet \overline{\bullet} \binom{n}{3} + \frac{2}{n} \bullet \overline{\bullet} \binom{n}{3} + \frac{3}{n} \bullet \overline{\bullet} \binom{n}{3}$$

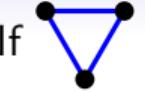
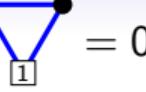


\bullet = probability of choosing an edge ... $|E|/\binom{n}{2}$



\bullet = probability of choosing an triple ... $\# \bullet \overline{\bullet} / \binom{n}{3}$

PROOF RECAP

If  =  = 0 then  $\leq 1/2$.

$$n^2|E| \geq \sum_{i \in V} 1^2 \cdot \sum_{i \in V} d_i^2 \geq \left(\sum_{i \in V} 1 \cdot d_i\right)^2 = 4|E|^2$$

$$\frac{1}{2} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \geq \frac{1}{3} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}^2 \geq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^2$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{3} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{2}{3} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{3}{3} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{n} \sum_{\substack{1 \\ \square \in V}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} f^2 \\ g^2 \end{bmatrix} \cdot \begin{bmatrix} f^2 \\ g^2 \end{bmatrix} \geq \begin{bmatrix} f \\ g \end{bmatrix}^2$$

DIFFERENT PROOF OF $\triangle = 0$ IMPLIES $\square \leq 1/2$

$$\square = \frac{1}{3} \cdot \cdot \cdot + \frac{2}{3} \cdot \cdot \cdot + \frac{3}{3} \cdot \cdot \cdot$$

DIFFERENT PROOF OF $\triangle = 0$ IMPLIES $\square \leq 1/2$

$$\begin{aligned}\square &= \frac{1}{3} \begin{array}{c} \bullet \\ \square \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ \square \\ \bullet \\ \backslash \\ \bullet \end{array} + \frac{3}{3} \begin{array}{c} \bullet \\ \square \\ \bullet \\ \backslash \\ \bullet \end{array} \\ &\leq \frac{1}{3} \begin{array}{c} \bullet \\ \square \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ \square \\ \bullet \\ \backslash \\ \bullet \end{array} + \frac{1}{2} \left[\left(\begin{array}{c} \bullet \\ \square \\ \bullet \end{array} - \begin{array}{c} \bullet \\ \square \\ \bullet \end{array} \right)^2 \right]\end{aligned}$$

DIFFERENT PROOF OF $\triangle = 0$ IMPLIES $\square \leq 1/2$

$$\begin{aligned}
 \square &= \frac{1}{3} \begin{array}{c} \bullet \\ \square \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ \square \\ \bullet \\ \diagup \\ \bullet \end{array} + \frac{3}{3} \begin{array}{c} \bullet \\ \square \\ \bullet \\ \diagup \\ \bullet \\ \diagdown \\ \bullet \end{array} \\
 &\leq \frac{1}{3} \begin{array}{c} \bullet \\ \square \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ \square \\ \bullet \\ \diagup \\ \bullet \end{array} + \frac{1}{2} \left[\left(\begin{array}{c} \bullet \\ \square \\ \bullet \end{array} - \begin{array}{c} \bullet \\ \square \\ \bullet \end{array} \right)^2 \right] \\
 &= \frac{1}{3} \begin{array}{c} \bullet \\ \square \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ \square \\ \bullet \\ \diagup \\ \bullet \end{array} + \frac{1}{2} \left[\begin{array}{c} \bullet \\ \square \\ \bullet \\ \diagup \\ \bullet \\ \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \square \\ \bullet \\ \diagup \\ \bullet \\ \diagdown \\ \bullet \end{array} - \begin{array}{c} \bullet \\ \square \\ \bullet \\ \diagup \\ \bullet \end{array} - \begin{array}{c} \bullet \\ \square \\ \bullet \\ \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \square \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \square \\ \bullet \end{array} \right]
 \end{aligned}$$

DIFFERENT PROOF OF $\begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} = 0$ IMPLIES $\begin{array}{c} \bullet \\ \bullet \end{array} \leq 1/2$

$$\begin{aligned}
 \begin{array}{c} \bullet \\ \bullet \end{array} &= \frac{1}{3} \begin{array}{c} \bullet & \bullet \\ & \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet & \bullet \\ & \bullet \end{array} + \frac{3}{3} \begin{array}{c} \bullet & \bullet \\ & \bullet \end{array} \\
 &\leq \frac{1}{3} \begin{array}{c} \bullet & \bullet \\ & \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet & \bullet \\ & \bullet \end{array} + \frac{1}{2} \left[\left(\begin{array}{c} \bullet \\ \boxed{1} \end{array} - \begin{array}{c} \bullet \\ \boxed{1} \end{array} \right)^2 \right] \\
 &= \frac{1}{3} \begin{array}{c} \bullet & \bullet \\ & \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet & \bullet \\ & \bullet \end{array} + \frac{1}{2} \left[\begin{array}{c} \bullet & \bullet \\ \boxed{1} & \bullet \end{array} + \begin{array}{c} \bullet & \bullet \\ \bullet & \boxed{1} \end{array} - \begin{array}{c} \bullet \\ \boxed{1} \end{array} - \begin{array}{c} \bullet \\ \boxed{1} \end{array} + \begin{array}{c} \bullet \\ \boxed{1} \end{array} + \begin{array}{c} \bullet \\ \boxed{1} \end{array} \right] \\
 &= \frac{1}{3} \begin{array}{c} \bullet & \bullet \\ & \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet & \bullet \\ & \bullet \end{array} + \frac{1}{2} \left(\frac{1}{3} \begin{array}{c} \bullet & \bullet \\ & \bullet \end{array} - \frac{2}{3} \begin{array}{c} \bullet & \bullet \\ & \bullet \end{array} - \frac{2}{3} \begin{array}{c} \bullet & \bullet \\ & \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet & \bullet \\ & \bullet \end{array} \right)
 \end{aligned}$$

DIFFERENT PROOF OF $\begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} = 0$ IMPLIES $\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \leq 1/2$

$$\begin{aligned}
 \begin{array}{c} \bullet \\ | \\ \bullet \end{array} &= \frac{1}{3} \begin{array}{c} \bullet \\ \bullet \\ | \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} + \frac{3}{3} \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \\
 &\leq \frac{1}{3} \begin{array}{c} \bullet \\ \bullet \\ | \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} + \frac{1}{2} \left[\left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} - \begin{array}{c} \bullet \\ | \\ \square \end{array} \right)^2 \right] \\
 &= \frac{1}{3} \begin{array}{c} \bullet \\ \bullet \\ | \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} + \frac{1}{2} \left[\begin{array}{c} \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \bullet \\ | \end{array} - \begin{array}{c} \bullet \\ | \\ \bullet \end{array} - \begin{array}{c} \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right] \\
 &= \frac{1}{3} \begin{array}{c} \bullet \\ \bullet \\ | \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} + \frac{1}{2} \left(\frac{1}{3} \begin{array}{c} \bullet \\ \bullet \\ | \end{array} - \frac{2}{3} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} - \frac{2}{3} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \\ \bullet \\ | \end{array} \right) \\
 &= \frac{1}{2} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} + \frac{1}{6} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} + \frac{1}{2} \begin{array}{c} \bullet \\ | \\ \bullet \end{array}
 \end{aligned}$$

DIFFERENT PROOF OF $\begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} = 0$ IMPLIES $\begin{array}{c} \bullet \\ \bullet \end{array} \leq 1/2$

$$\begin{aligned}
 \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} &= \frac{1}{3} \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} + \frac{3}{3} \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \\
 &\leq \frac{1}{3} \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} + \frac{1}{2} \left[\left(\begin{array}{c} \bullet \\ \bullet \\ \boxed{1} \end{array} - \begin{array}{c} \bullet \\ \bullet \\ \boxed{1} \end{array} \right)^2 \right] \\
 &= \frac{1}{3} \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} + \frac{1}{2} \left[\begin{array}{c} \bullet \\ \bullet \\ \boxed{1} \end{array} + \begin{array}{c} \bullet \\ \bullet \\ \boxed{1} \end{array} - \begin{array}{c} \bullet \\ \bullet \\ \boxed{1} \end{array} - \begin{array}{c} \bullet \\ \bullet \\ \boxed{1} \end{array} + \begin{array}{c} \bullet \\ \bullet \\ \boxed{1} \end{array} + \begin{array}{c} \bullet \\ \bullet \\ \boxed{1} \end{array} \right] \\
 &= \frac{1}{3} \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} + \frac{1}{2} \left(\frac{1}{3} \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} - \frac{2}{3} \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} - \frac{2}{3} \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \right) \\
 &= \frac{1}{2} \begin{array}{c} \bullet \\ \bullet \end{array} + \frac{1}{6} \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} + \frac{1}{2} \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \\
 &\leq \frac{1}{2} \underbrace{\left(\begin{array}{c} \bullet \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \right)}_{=1} = \frac{1}{2}
 \end{aligned}$$

AUTOMATED APPROACH

$$\begin{aligned} \bullet \bullet &\leq \bullet \bullet + \left\| \left(\begin{array}{c} \bullet \\ \square \\ \end{array} - \begin{array}{c} \bullet \\ \square \\ \end{array} \right)^2 \right\| = \frac{1}{2} \bullet \bullet + \frac{1}{6} \bullet \bullet + \frac{1}{2} \bullet \bullet \\ &\leq \frac{1}{2} \left(\bullet \bullet + \bullet \bullet + \bullet \bullet \right) = \frac{1}{2} \end{aligned}$$

In general as sum of squares

$$f \leq f + \sum_h \left\| h \right\|^2 = \sum_{G \in \mathcal{F}_n} c_G \cdot G \leq \max_{G \in \mathcal{F}_n} c_G$$

f, g linear combination of flags

$\mathcal{F}_n \dots$ flags on n vertices

SOS proofs can be optimized by semidefinite programming

SUM OF SQUARES

$$\left\| \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \right\| \leq \left\| \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \hline 1 \end{array} \right\| + \left\| \left(\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \hline 1 \end{array} - \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \hline 1 \end{array} \right)^2 \right\| = \frac{1}{2} \bullet \bullet + \frac{1}{6} \bullet \bullet + \frac{1}{2} \bullet \bullet \leq \frac{1}{2}$$

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Semidefinite matrix

$$\left\| \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \right\| \leq \left\| \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \hline 1 \end{array} \right\| + \left\| \left(\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \hline 1 \end{array}, \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \hline 1 \end{array} \right) \underbrace{\left(\begin{array}{cc} a & c \\ c & b \end{array} \right)}_{=M \succeq 0} \left(\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \hline 1 \end{array}, \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \hline 1 \end{array} \right)^T \right\| = \sum_{G \in \mathcal{F}_n} c_{G,M} \cdot G \leq \max_{G \in \mathcal{F}_n} c_{G,M}$$

SUM OF SQUARES

$$\left\| \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \right\| \leq \left\| \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \right\| + \left\| \left(\begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} - \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \right)^2 \right\| = \frac{1}{2} \bullet \bullet + \frac{1}{6} \bullet \bullet + \frac{1}{2} \bullet \bullet \leq \frac{1}{2}$$

In general as sum of squares

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Flag Algebras and Its Application

- Flag Algebras

- Sum of squares

Make the subscript of Max under

SUM OF SQUARES

$$\begin{array}{c} \text{I} \leq \text{I} + \left[\left(\begin{array}{c} \text{I} \\ \text{I} \\ \hline \text{I} & \text{I} \end{array} - \begin{array}{c} \text{I} \\ \text{I} \\ \hline \text{I} & \text{I} \end{array} \right)^2 \right] = \frac{1}{2} \begin{array}{c} \text{I} \\ \text{I} \\ \hline \text{I} & \text{I} \end{array} + \frac{1}{6} \begin{array}{c} \text{I} \\ \text{I} \\ \text{I} \\ \hline \text{I} & \text{I} & \text{I} \end{array} + \frac{1}{2} \begin{array}{c} \text{V} \\ \text{V} \\ \hline \text{V} & \text{V} \end{array} \leq \frac{1}{2} \end{array}$$

In general as sum of squares

$$f \leq f + \sum_h \|h^2\| = \sum_{G \in \mathcal{F}_n} c_G \cdot G \leq \max_{G \in \mathcal{F}_n} c_G$$

Semidefinite matrix

$$\begin{array}{c} \text{I} \leq \text{I} + \left[\left(\begin{array}{c} \text{I} \\ \text{I} \\ \hline \text{I} & \text{I} \end{array} - \begin{array}{c} a & c \\ c & b \\ \hline \text{I} & \text{I} \end{array} \right) \left(\begin{array}{c} a & c \\ c & b \\ \hline \text{I} & \text{I} \end{array} \right)^T \right] = \sum_{G \in \mathcal{F}_n} c_{G,M} \cdot G \leq \min_{M \geq 0} \max_{G \in \mathcal{F}_n} c_{G,M} \end{array}$$

Rainbow Triangles

<https://arxiv.org/abs/2511.21061>

JOINTS

Joint in \mathbb{R}^d is a point where d lines that span \mathbb{R}^d intersect.

What is the maximum number of joints for N lines?

THEOREM (CHAO AND HANS YU 2023+)

Number of joints is maximized by k hyperplanes whose intersection give $N = \binom{k}{d-1}$ lines and $\binom{k}{d}$ joints.

Assymptotically by Hans Yu and Zhao 2023.

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Multijoint problem: In \mathbb{R}^3 , lines of three colors, maximize rainbow joints.

hyperplane \rightarrow vertex

intersection of hyperplanes \rightarrow edge

joint \rightarrow rainbow triangle

THEOREM (CHAO AND HANS YU 2024+)

In 3-edge colored graph $\left(\# \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \right)^2 \leq 2 \left(\# \begin{array}{c} \bullet \\ \mid \\ \bullet \end{array} \right) \cdot \left(\# \begin{array}{c} \bullet \\ \mid \\ \bullet \end{array} \right) \cdot \left(\# \begin{array}{c} \bullet \\ \mid \\ \bullet \end{array} \right)$.

JOINTS

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$$\text{In 3-edge colored graph } \left(\# \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \right)^2 \leq 2 \left(\# \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right) \cdot \left(\# \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right) \cdot \left(\# \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right).$$

In flag algebras

$$\left(\# \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \right)^2 \leq 9 \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \cdot \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \cdot \begin{array}{c} \bullet \\ | \\ \bullet \end{array}$$

Automated sum-of-squares proof needs 540GB RAM

Flag Algebras and Its Application

└ Flag Algebras

└ Joints

Motivation is for joins - one take lined colored reg, green and blue and asks for rainbow lines.
Their proof uses entropy method.

1,601,952 configurations and 540GB ram

JOINTS

Multijoint problem: In \mathbb{R}^3 , lines of three colors, maximize rainbow joints.
hyperplane \rightarrow vertex
intersection of hyperplanes \rightarrow edge
joint \rightarrow rainbow triangle

THEOREM (CHAO AND HANS YU 2024+)

In 3-edge colored graph $(\# \text{ } \textcolor{blue}{V})^2 \leq 2 \left(\# \text{ } \textcolor{blue}{I} \right) \cdot \left(\# \text{ } \textcolor{blue}{E} \right) \cdot \left(\# \text{ } \textcolor{blue}{J} \right)$.

In flag algebras

$$(\textcolor{blue}{V})^2 \leq 9 \text{ } \textcolor{blue}{I} \text{ } \textcolor{blue}{E} \text{ } \textcolor{blue}{J}$$

Automated sum-of-squares proof needs 540GB RAM

$$\begin{array}{c} \text{[Diagram: Two vertical lines with dots at the top and bottom, representing a 2x2 matrix.]} \\ \times \end{array} \geq \frac{1}{3} \cdot \left(\begin{array}{c} \text{[Diagram: A 2x2 matrix with green dashed lines and blue solid lines.]} \\ + \end{array} \begin{array}{c} \text{[Diagram: A 2x2 matrix with green dashed lines and red solid lines.]} \\ + \end{array} \begin{array}{c} \text{[Diagram: A 2x2 matrix with green dashed lines and blue solid lines.]} \\ + \end{array} \right) = 4 \cdot \left[\begin{array}{c} \text{[Diagram: A 2x2 matrix with red solid lines and a red bracket below it labeled '1'.]} \\ \text{[Diagram: A 2x2 matrix with green dashed lines and a green bracket above it labeled '2'.]} \end{array} \right]^2.$$

$$\begin{array}{c} \text{Diagram: Two vertical columns of two nodes each, connected by a dashed green line between the top nodes and a solid blue line between the bottom nodes.} \\ \times \geq \frac{1}{3} \cdot \left(\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} \right) = 4 \cdot \left[\left[\text{Diagram 1} \right]^2 \right]. \end{array}$$

$$\begin{aligned} & \text{Diagram 1} = 6 \cdot \left[\left[\text{Diagram 1} \right] \right] = 6 \cdot \left[\left[\text{Diagram 1} \right] \times \left[\left[\text{Diagram 1} \right] \right] \right] \\ & \leq 6 \cdot \sqrt{\left[\left[\left(\text{Diagram 1} \right)^2 \right] \right]} \cdot \sqrt{\left[\left[\left(\text{Diagram 1} \right)^2 \right] \right]} \\ & = 6 \cdot \sqrt{\frac{1}{12} \left(\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} \right)} \cdot \sqrt{\text{Diagram 1}} \\ & \leq 3 \cdot \sqrt{\text{Diagram 1}} \cdot \sqrt{\text{Diagram 1}} = 3 \cdot \sqrt{\text{Diagram 1}} \times \sqrt{\text{Diagram 1}}. \end{aligned}$$

Flag Algebras and Its Application

└ Flag Algebras

└ Balogh, Bradshaw, Garcia, L. 2025+

$$\begin{aligned} \text{I} &\geq \frac{1}{3} \cdot (\text{II} + \text{III} + \text{IV} + \text{V}) = 4 \cdot [\text{V}^{\text{II}}]. \\ \text{V} &= 6 \cdot [\text{V}^{\text{II}}] = 6 \cdot [\text{V}^{\text{II}} \times \text{I}] \\ &\leq 6 \cdot \sqrt{([\text{V}^{\text{II}}]^2)} \cdot \sqrt{([\text{I}]^2)} \\ &= 6 \cdot \sqrt{\frac{1}{12} (\text{II} + \text{III} + \text{IV} + \text{V})} \cdot \sqrt{1} \\ &\leq 3 \cdot \sqrt{\text{I}} \times \sqrt{\text{I}} = 3 \cdot \sqrt{\text{I}} \times \sqrt{\text{I}} \end{aligned}$$

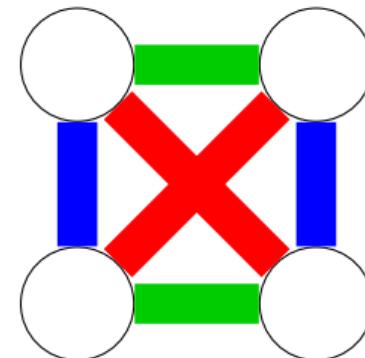
Can be written without flags, a simple counting proof.

THEOREM (CHAO AND HANS YU 2024+)

$$\text{Diagram: A V-shaped structure with two red wavy lines and a blue vertical line. To its right is a 3x3 grid of points with vertical and horizontal dashed lines connecting them.}$$
$$\leq 3\sqrt{\dots}$$

THEOREM (BALOGH, BRADSHAW, GARCIA, L.
2025+)

$$\text{Diagram: A 2x2 grid of points with red and green wavy lines forming an X. To its right is a 2x2 grid of circles with a red X and blue vertical lines.}$$
$$\leq \frac{3}{2} \cdot \left(\dots \times \dots \times \dots \right)^{2/3}$$



We also have exactness and translation to counting and a short entropy proofs.

Flag Algebras and Its Application

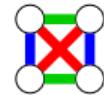
└ Flag Algebras

THEOREM (CHAO AND HANS YU 2024+)

$$\text{V} \leq 3\sqrt{1+1}$$

THEOREM (BALOGH, BRADSHAW, GARCIA, L. 2025+)

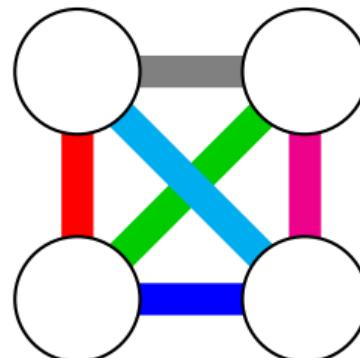
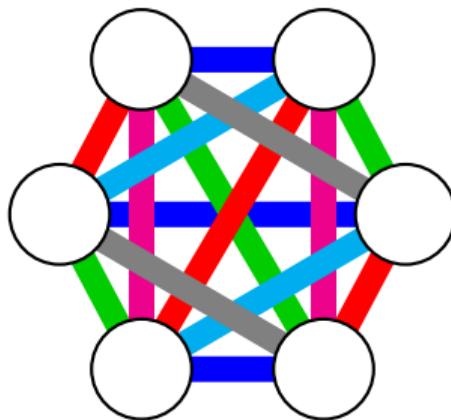
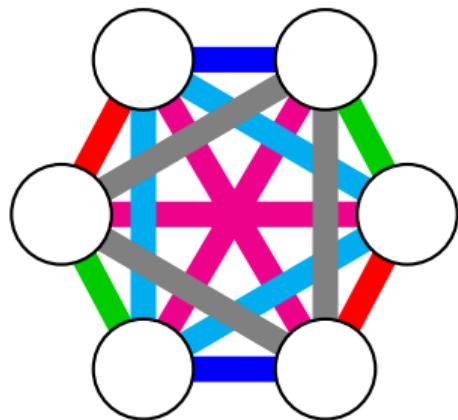
$$\text{X} \leq \frac{13}{n^2} \left(\text{V} \times \text{V} \times \text{V} \right)^{2/3}$$



We also have exactness and translation to counting and a short entropy proofs.

We also have exactness results.

FURTHER DIRECTIONS



QUESTION

Let G be a graph with edges colored by colors $\{1, \dots, 6\}$. Denote by C_i the number of edges colored by color i . Let H be the number of rainbow copies of K_4 in G . Is it true that $H \leq \sqrt[3]{\prod_i C_i}$?

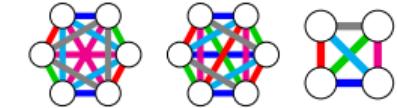
Flag Algebras and Its Application

└ Flag Algebras

└ Further Directions

We note how to do it for a fixed rainbow coloring

FURTHER DIRECTIONS



QUESTION

Let G be a graph with edges colored by colors $\{1, \dots, 6\}$. Denote by C_i the number of edges colored by color i . Let H be the number of rainbow copies of K_4 in G . Is it true that $H \leq \sqrt[6]{\prod_i C_i}^7$?

Mathematical Biology

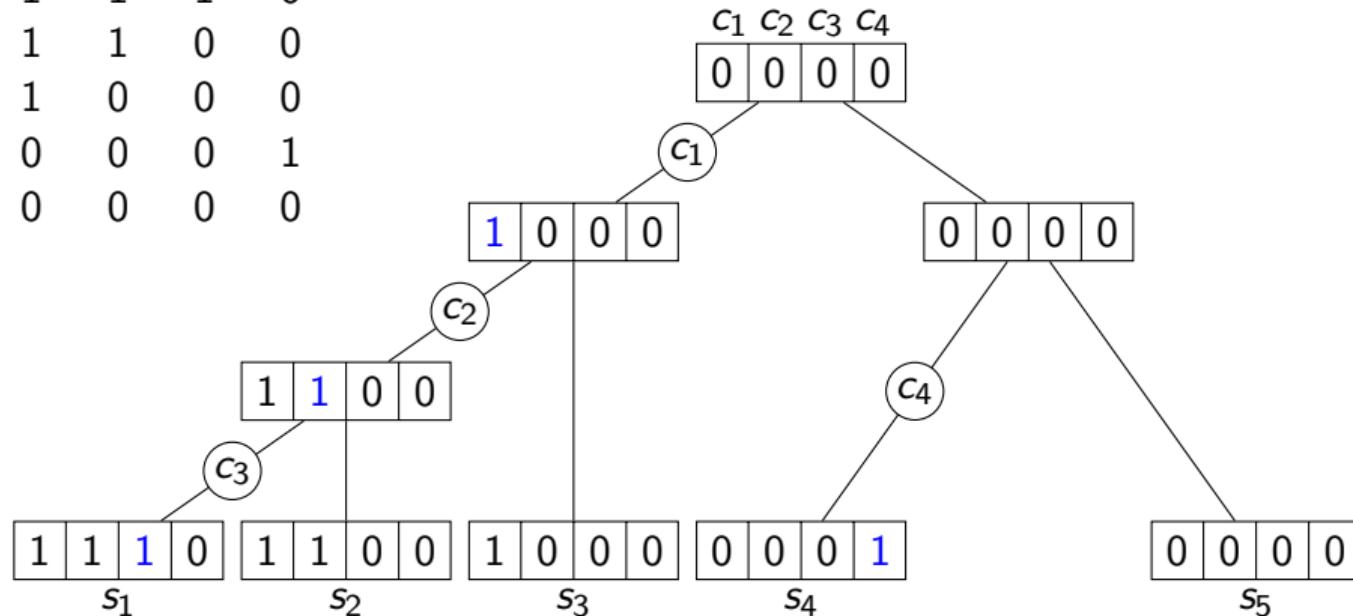
PHYLOGENY

How do we reconstruct an evolutionary history (*phylogeny*) from observations of living species from characters?

	Spine	Fur	Fins	Wings
Seal	1	1	1	0
Dog	1	1	0	0
Lizard	1	0	0	0
Butterfly	0	0	0	1
Worm	0	0	0	0

PHYLOGENY

	c_1	c_2	c_3	c_4
s_1	1	1	1	0
s_2	1	1	0	0
s_3	1	0	0	0
s_4	0	0	0	1
s_5	0	0	0	0



ASSUMPTIONS AND PROBLEMS

Perfect phylogeny model is fundamental, but inaccurate

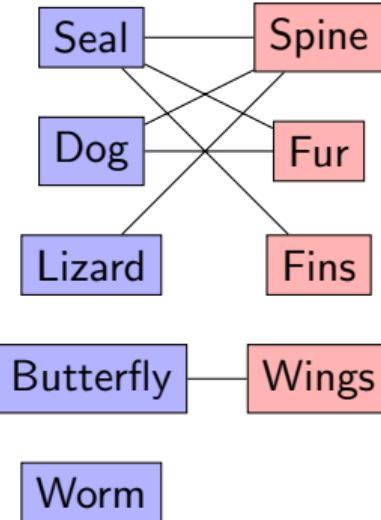
LEMMA (THREE-GAMETE CONDITION, HUDSON-KAPLAN '85)

A set of species and characters has a perfect phylogeny if and only if for every pair of traits, no three species present all of the combinations 10, 01, 11.

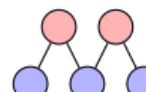
	Spine	Fins		Spine	Wings
Seal	1	1	Bird	1	1
Lizard	1	0	Lizard	1	0
Butterfly	0	0	Butterfly	0	1

INCIDENCE GRAPH

	Spine	Fur	Fins	Wings
Seal	1	1	1	0
Dog	1	1	0	0
Lizard	1	0	0	0
Butterfly	0	0	0	1
Worm	0	0	0	0

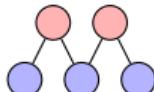


CLAIM

The incidence graph contains no induced copies of  if and only if the three-gamete condition is satisfied

MAIN PROBLEM

PROBLEM

How many induced copies of $M =$  can we possibly have?

Counting M measures how far from perfect.

- Inducibility problem in *red-blue graphs*
- Bipartite graphs with fixed two-colorings
- Isomorphisms are graph isomorphisms and preserve colors

RESULTS

THEOREM (EULENSTEIN, HALFPAP, L., MIYASAKI, PFENDER, VOLEC 2025+)

Fix $\alpha > 0$. Let $G_{r,b}$ be a red-blue graph with r red vertices and b blue vertices with $\frac{r}{b} = \alpha$. Then

$$\# \begin{array}{c} \text{red} \\ \text{blue} \\ \text{blue} \\ \text{blue} \end{array} \leq \frac{r^2 b^3}{81} + o(r^2 b^3)$$

COROLLARY (EHLMPV)

If G_n is a red-blue graph on n vertices then

$$\# \begin{array}{c} \text{red} \\ \text{blue} \\ \text{blue} \\ \text{blue} \end{array} \leq \frac{2^2 3^4 n^5}{15^5} + o(n^5).$$

ASYMPTOTIC EXTREMAL EXAMPLES

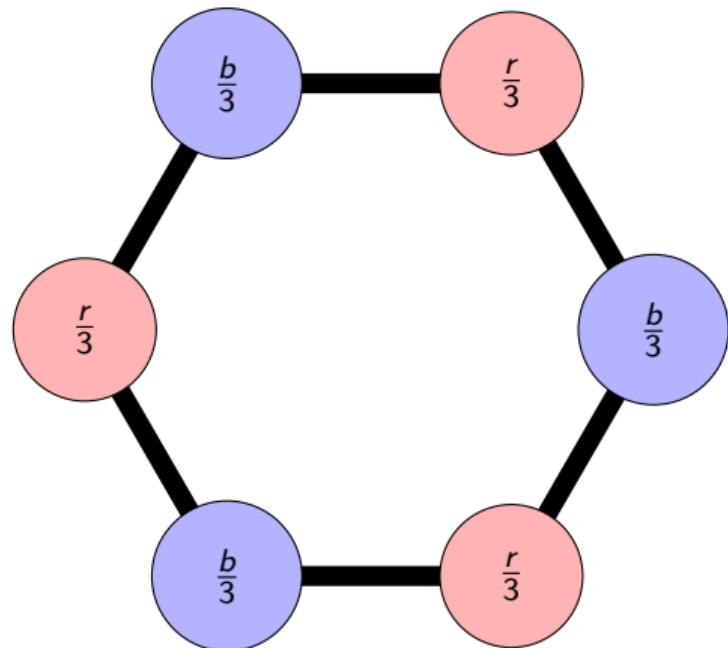


FIGURE: $C_6(r, b)$

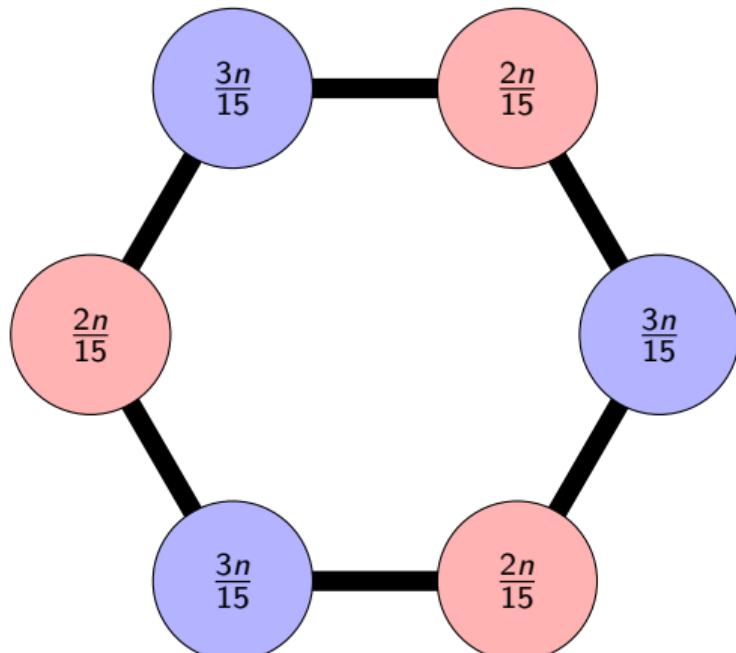
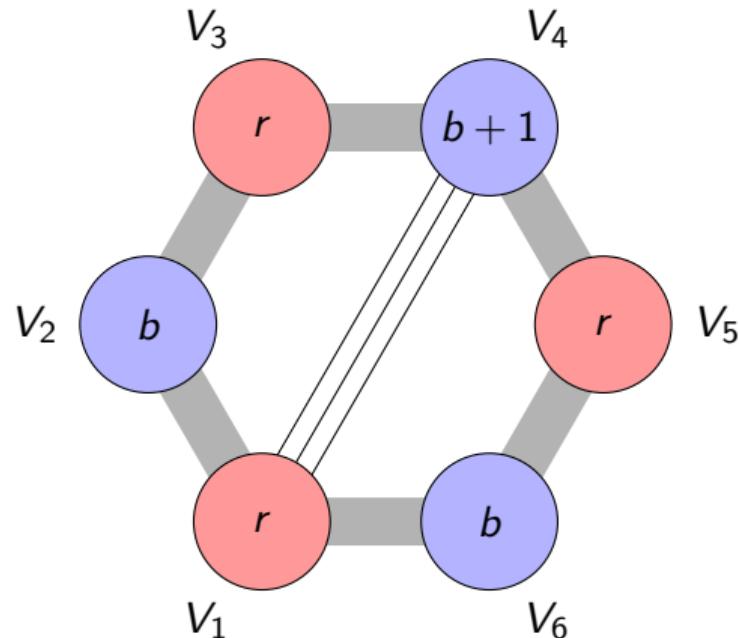


FIGURE: $C_6(n)$

FUTURE DIRECTIONS

- Determine lower-order terms and stronger characterizations



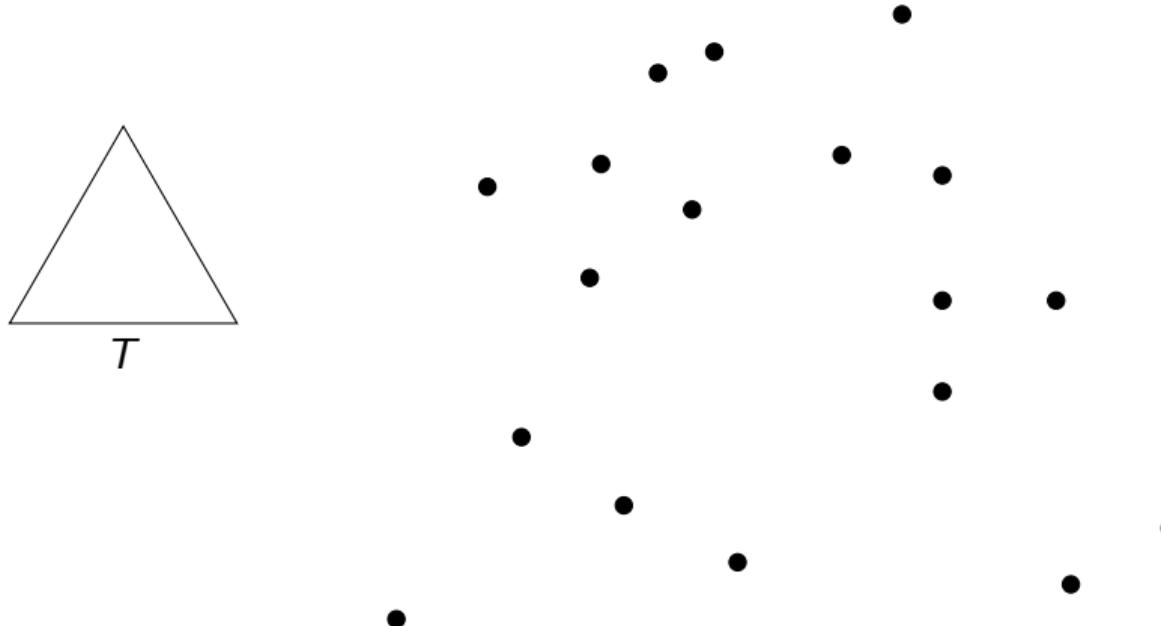
ε -similar Triangles

<https://arxiv.org/abs/2101.10304>

PROBLEM

Let T be a triangle and $n \in \mathbb{N}$ fixed.

Which n points in \mathbb{R}^2 maximize the number of triangles similar to T ?

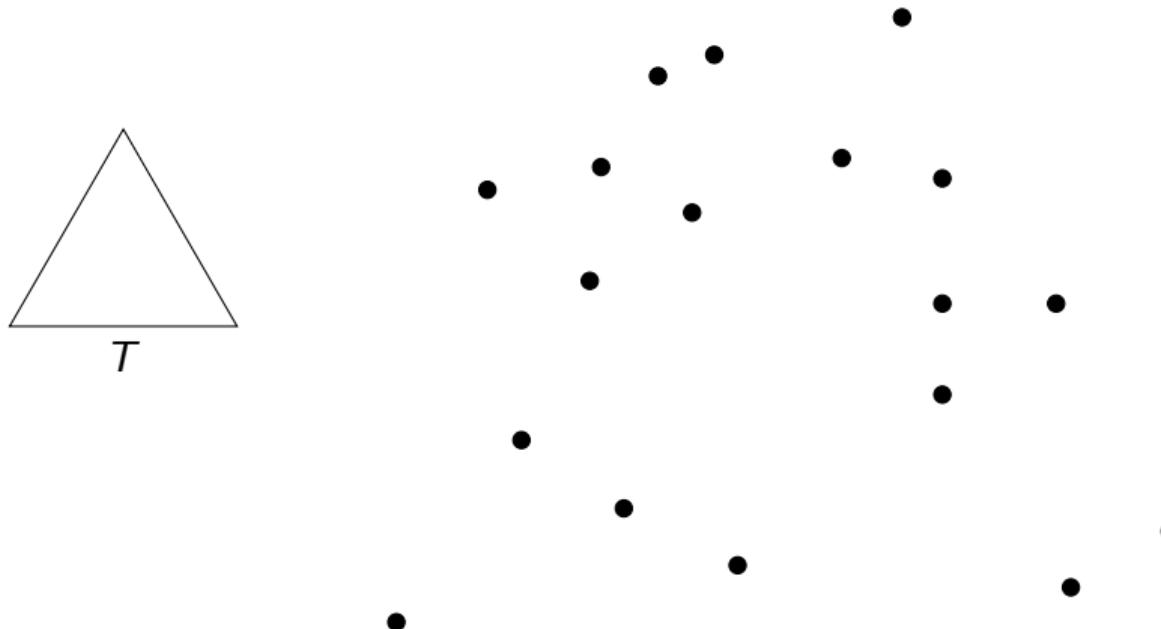


T_1 and T_2 are *ε -similar* if their inner angles differ by at most ε .
(OK to move, scale, rotate, ε -perturb)

PROBLEM

Let T be a triangle and $n \in \mathbb{N}$ fixed. (and $\varepsilon > 0$ fixed)

Which n points in \mathbb{R}^2 maximize the number of triangles similar to T ?

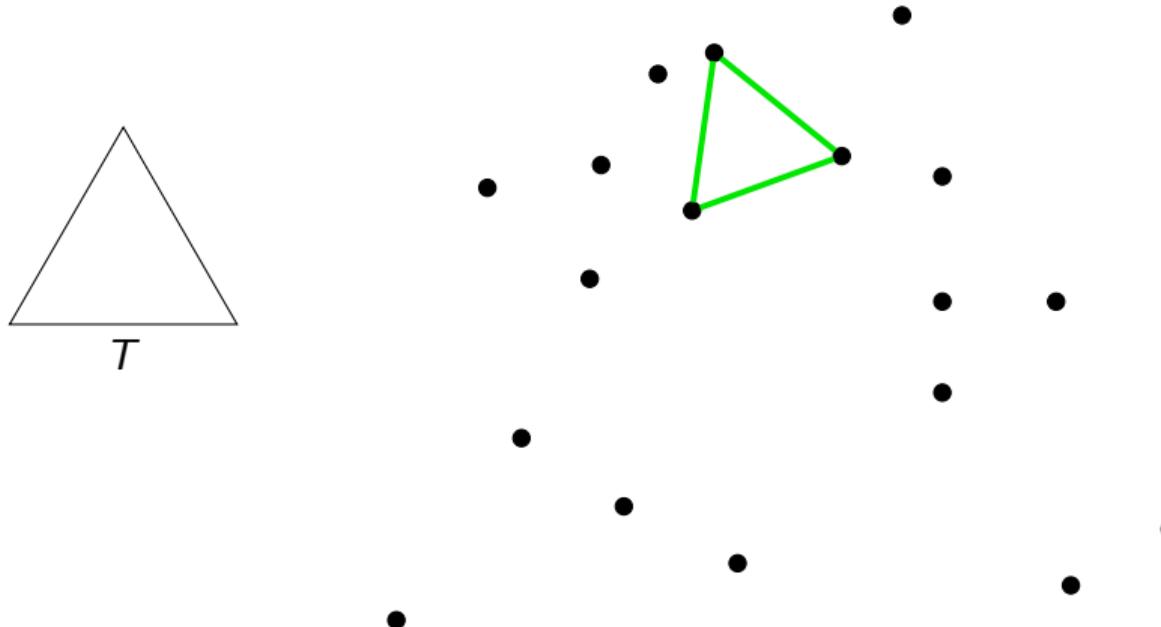


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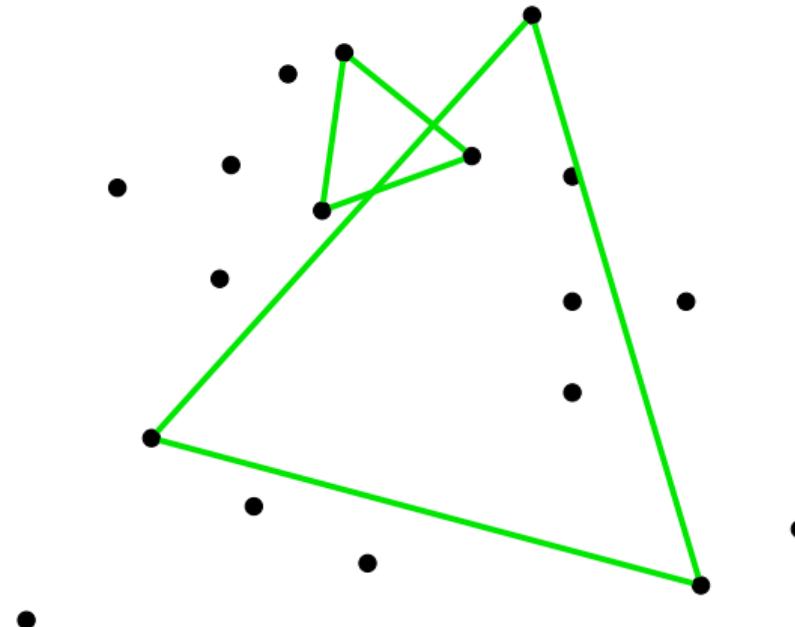
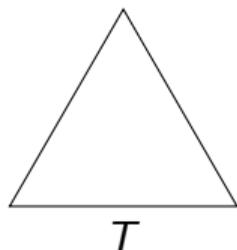


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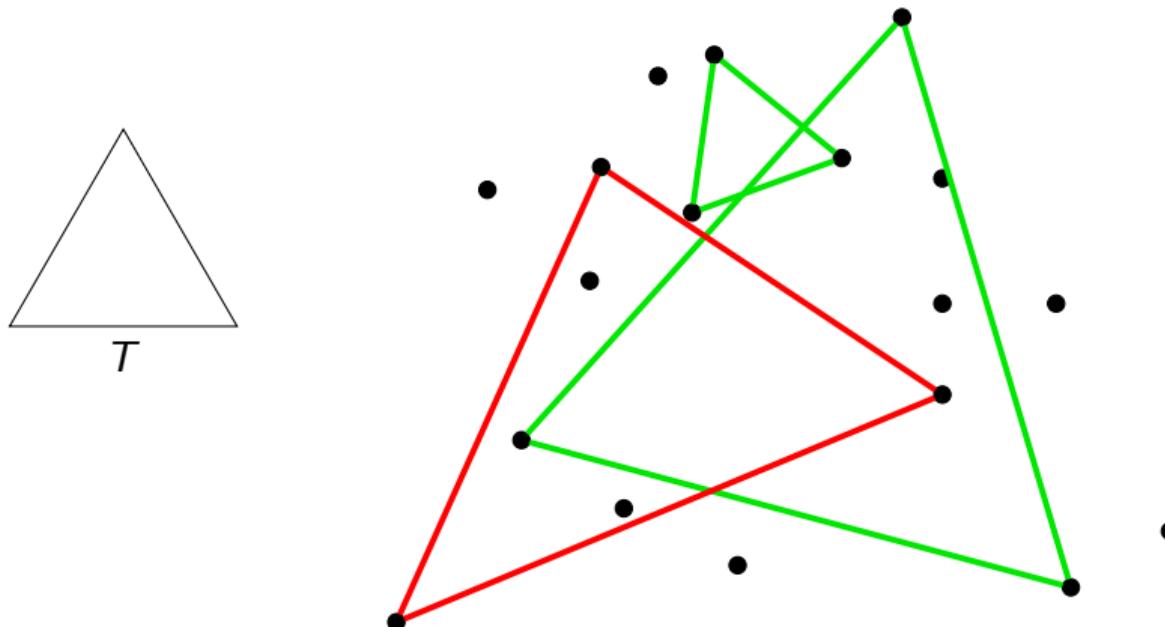


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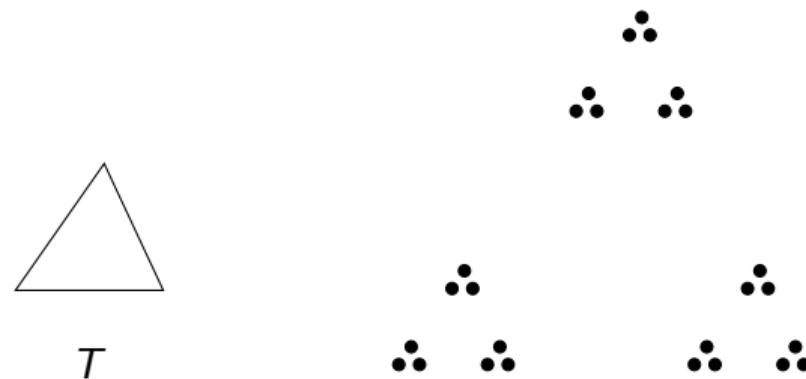


T_1 and T_2 are *ε -similar* if their inner angles differ by at most ε .
(OK to move, scale, rotate, ε -perturb)

LOWER BOUND CONSTRUCTION

Let T be a triangle and $n \in \mathbb{N}$ fixed. (and $\varepsilon > 0$ fixed)

Which n points in \mathbb{R}^2 maximize the number of triangles similar to T ?



$h(n, T, \varepsilon) := \max \# \text{ of } \varepsilon\text{-similar triangles to } T$, it is at least $\frac{1}{4} \binom{n}{3} (1 + o(1))$.

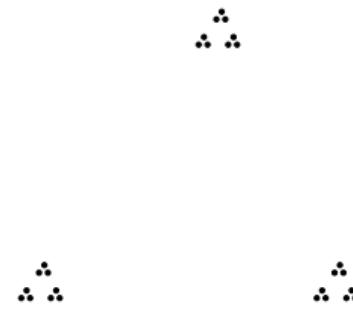
LOWER BOUND CONSTRUCTION

Let T be a triangle and $n \in \mathbb{N}$ fixed. (and $\varepsilon > 0$ fixed)

Which n points in \mathbb{R}^2 maximize the number of triangles similar to T ?



T



$h(n, T, \varepsilon) := \max \# \text{ of } \varepsilon\text{-similar triangles to } T$, it is at least $\frac{1}{4} \binom{n}{3} (1 + o(1))$.

RESULTS

THEOREM (BÁRÁNY AND FÜREDI (2019))

For almost every triangle T there is an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$,

$$h(n, T, \varepsilon) \leq 0.25072 \binom{n}{3} (1 + o(1)).$$

If T is equilateral, then $h(n, T, \varepsilon) = \frac{1}{4} \binom{n}{3} (1 + o(1))$

THEOREM (BALOGH, CLEMEN, L. (2022))

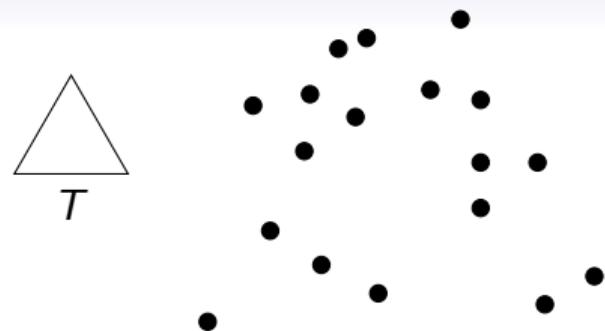
For almost every triangle T there is an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$,

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$h(n, T, \varepsilon) := \max \# \text{ of } \varepsilon\text{-similar triangles to } T, \text{ it is at least } \frac{1}{4} \binom{n}{3} (1 + o(1)).$

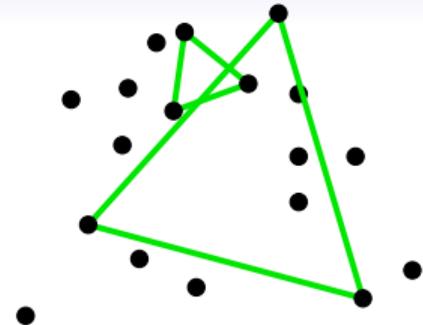
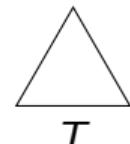
Let T and ε are given

- Fix n points in the plane.
- For every T' ε -similar to T , add a 3-edge
- Investigate the resulting hypergraph H
 H has no subhypergraph in $\mathcal{F} = \{K_4^3, \dots\}$



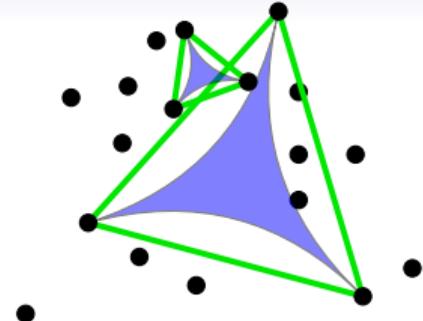
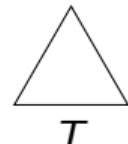
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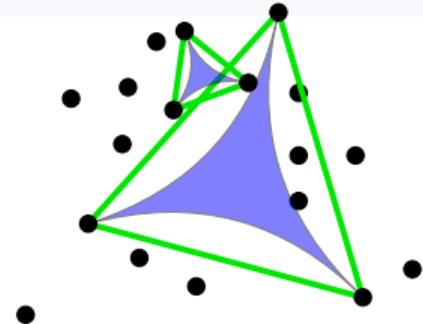
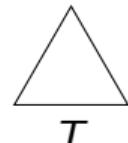
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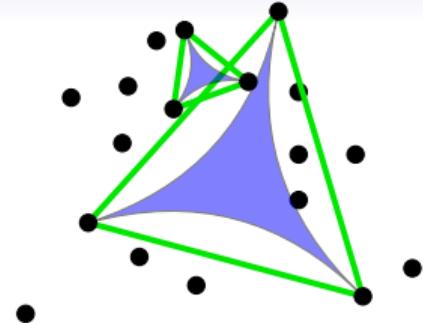
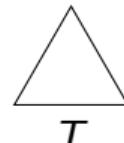


THEOREM (BALOGH, CLEMEN, L. (2022))

\mathcal{F} -free hypergraph has at most $\frac{1}{4} \binom{n}{3} (1 + o(1))$ edges.

Let T and ε are given

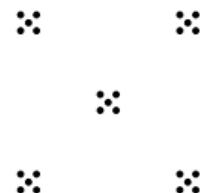
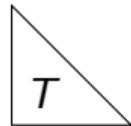
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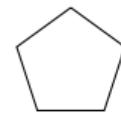
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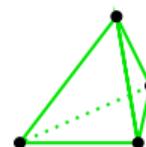
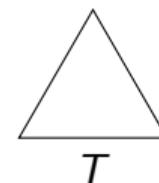
All triangles?



Other Shapes?



in \mathbb{R}^d ?



Counting k -SAT functions

QUESTION

Count functions

$$f : \{0, 1\}^n \rightarrow \{0, 1\} \quad 2^{2^n}$$

k-SAT FUNCTION can be defined as

$$f(x_1, \dots, x_n) = C_1 \vee C_2 \vee \dots \vee C_m$$

$$C_i = \underbrace{z_1 \wedge z_2 \wedge \dots \wedge z_k}_{\text{all different variables}} \quad z_i \in \{x_1, \neg x_1, x_2, \neg x_2, \dots, x_n, \neg x_n\}$$

x_i variable, C_i clause, z_i literal

example $k = 3$

$$x_1 \wedge x_2 \rightarrow (x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge \neg x_3)$$

$$x_1 \wedge x_2 \wedge \neg x_2 \rightarrow \text{always false}$$

Every k -SAT function has a formula but the formula may not be unique.

number of $f : \{0, 1\}^n \rightarrow \{0, 1\}$ 2^{2^n}

number of k -SAT formula $2^{2^k \binom{n}{k}}$

number of k -SAT functions?

k -SAT formula is *monotone* if it uses only x_1, x_2, \dots, x_n , (i.e. no $\neg x_i$ is used)

All monotone k -SAT formula give different functions

$$g \notin x_1 \wedge \dots \wedge x_k \ni f \quad f \neq g \text{ at } x_1 = \dots = x_k = 1, x_{k+1} = \dots = x_m = 0$$

Number of monotone k -SAT functions $2^{\binom{n}{k}}$

k -SAT formula is *unate* if it uses at most one of $\{x_i, \neg x_i\} = \{x_i, \bar{x}_i\}$

Number of unate k -SAT functions $(1 + o(1))2^{n + \binom{n}{k}}$

Functions avoiding x_i counted multiple times

CONJECTURE (BOLLOBÁS, BRIGHTWELL, LEADER 2003)

Fix $k \geq 2$, $1 - o(1)$ fraction of k -SAT functions are unate as $n \rightarrow \infty$. $(1 + o(1))2^{n+{n \choose k}}$

- # 2-SAT functions is $2^{(1+o(1))\binom{n}{2}}$. Bollobás, Brightwell, Leader 2003 using Szemerédi regularity lemma
- Conjecture true for $k = 2$ Allen 2007 using Szemerédi regularity lemma
- Conjecture true for $k = 2$ Ilinca, Kahn 2009 without Szemerédi regularity lemma
- Conjecture true for $k = 3$ Ilinca, Kahn 2012 using hypergraph regularity lemma
- Conjecture true for $k = 4, 5$ Dong, Mani, Zhao 2022

Conjecture true for all k :-) Balogh, Dong, Lidický, Mani, Zhao

- Step 1: Reduction to a Turán type problem

Dong, Mani, Zhao using blow-up, saturation, container method

- Step 2: Solving the extremal problem

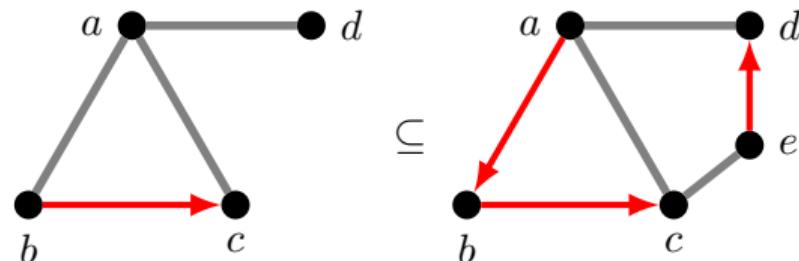
Balogh, Dong, L., Many, Zhao: computer free flag-algebra

DIRECTED HYERGRAPH TURÁN PROBLEM

Partially directed k-graph is a k -uniform hypergraph, where every edge is

- undirected
- rooted at one vertex (directed towards one vertex)

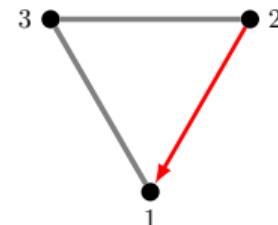
$\vec{H} \subseteq \vec{G}$ if \vec{H} could be obtained from \vec{G} by removing some vertices, edges, or orientations.



\vec{T}_k

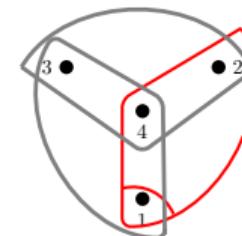
- $\vec{T}_2 = \{\hat{1}2, 13, 23\}$

① ② ③
-1- 2- ...
-1- 3-
..... 2- 3-



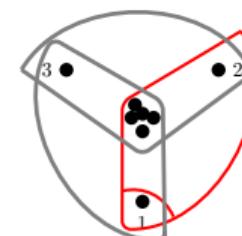
- $\vec{T}_3 = \{\hat{1}24, 134, 234\}$

① ② ③ ④
-1- 2- ... 4-
-1- 3- 4-
..... 2- 3- 4-



- $\vec{T}_k = \{\hat{1}24 \cdots k+1, 134 \cdots k+1, 234 \cdots k+1\}$

① ② ③ ④ ⑤
-1- 2- ... 4- 5-
-1- 3- 4- 5-
..... 2- 3- 4- 5-



EXTREMAL PROBLEM

G is k -uniform, n -vertex, \vec{T}_k -free.



$$\alpha := \frac{e_{\text{undirected}}(G)}{\binom{n}{k}}$$



$$\beta := \frac{e_{\text{directed}}(G)}{\binom{n}{k}}$$

Given k, θ , what is

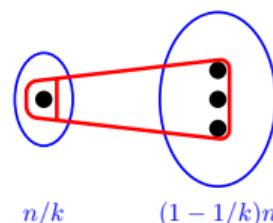
$$\max\{\alpha + \theta\beta\}?$$

Special (open) case:

Show $\alpha + \theta\beta \leq 1$ when $1 \leq \theta \leq \left(1 - \frac{1}{k}\right)^{1-k} \approx e$

Constructions:

Complete undirected graph



CONJECTURE (BOLLOBÁS, BRIGHTWELL, LEADER 2003)

Fix $k \geq 2$, almost all k -SAT functions are unate.

THEOREM (DONG, MANI, ZHAO)

If $\alpha + \theta\beta \leq 1$ for some $\theta > \log_2 3$ then almost all k -SAT functions are unate.

This theorem is a lot of work. Uses hypergraph containers by Balogh, Morris, Samotij; Saxton, Thomasson (AMS Steele prize 2024)

THEOREM (DONG, MANI, ZHAO)

Conjecture true for $k \leq 5$.

THEOREM (BALOGH, DONG, LIDICKÝ, MANI, ZHAO)

Conjecture true for all k .

PROOF FOR $k \geq 4$ USING FLAG ALGEBRAS

If graphs represent densities as

$$\bullet \underset{-1-}{\cdots} \bullet \underset{-2-}{\cdots} \bullet \underset{-3-}{\cdots} \bullet \underset{-4-}{\cdots} = \alpha := \frac{e_{\text{undirected}}(G)}{\binom{n}{k}}$$

$$\bullet \underset{-1-}{\cdots} \bullet \underset{-2-}{\cdots} \bullet \underset{-3-}{\cdots} \bullet \underset{-4-}{\cdots} = \beta := \frac{e_{\text{directed}}(G)}{\binom{n}{k}}$$

then

$$\begin{aligned}
 & \alpha + \theta\beta \\
 &= \bullet \underset{-1-}{\cdots} \bullet \underset{-2-}{\cdots} \bullet \underset{-3-}{\cdots} \bullet \underset{-4-}{\cdots} + \theta \bullet \underset{-1-}{\cdots} \bullet \underset{-2-}{\cdots} \bullet \underset{-3-}{\cdots} \bullet \underset{-4-}{\cdots} \\
 &\leq \bullet \underset{-1-}{\cdots} \bullet \underset{-2-}{\cdots} \bullet \underset{-3-}{\cdots} \bullet \underset{-4-}{\cdots} + \theta \bullet \underset{-1-}{\cdots} \bullet \underset{-2-}{\cdots} \bullet \underset{-3-}{\cdots} \bullet \underset{-4-}{\cdots} + \left[(a \underset{\boxed{1}}{\cdots} \underset{\boxed{2}}{\cdots} \underset{\boxed{3}}{\cdots} \bullet - b \underset{\boxed{1}}{\cdots} \underset{\boxed{2}}{\cdots} \underset{\boxed{3}}{\cdots} \bullet \underset{-4-}{\cdots})^2 \right] \\
 &\leq 1
 \end{aligned}$$

$$\text{for } \theta = 1 + \frac{1}{\sqrt{2}} \geq 1.707 > \log_2 3 \quad a = \frac{1}{\sqrt{2}}, b = \frac{k(\theta-1)-1}{\sqrt{2}} \quad k \geq 4$$

PROOF FOR $k = 2$ AND $k = 3$

$$1 \underset{-1-2-}{\bullet\bullet} + 1.7 \underset{-1-2-}{\bullet\bullet} + \left[\left[\left(-1 \underset{-1-}{\boxed{1}} \underset{-2-}{\bullet} - 1 \underset{-1-}{\boxed{1}} \underset{-2-}{\bullet} + 0.98 \underset{\square}{1} \bullet \right)^2 \right] \right] \leq 1$$

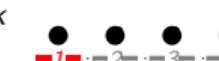
$$1 \underset{-1-2-3-}{\bullet\bullet\bullet} + 1.7 \underset{-1-2-3-}{\bullet\bullet\bullet} + 0.039 \times \left[\left[\left(-6 \underset{-1-2-3-}{\boxed{1}\boxed{2}} \underset{-3-}{\bullet} - 5 \underset{-1-2-3-}{\boxed{1}\boxed{2}} \underset{-3-}{\bullet} + 5 \underset{\square}{1} \underset{\square}{2} \bullet \right)^2 \right] \right] \leq 1$$

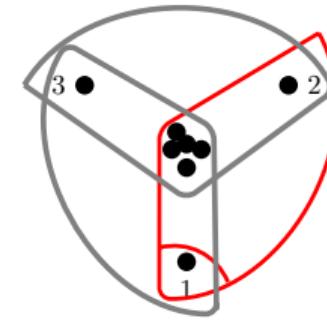
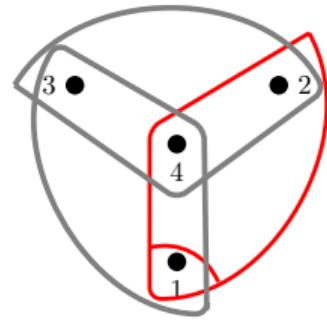
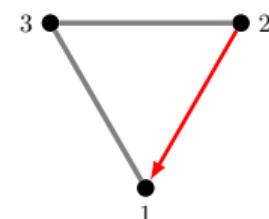
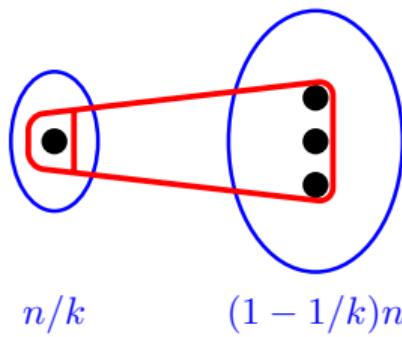
FUTURE DIRECTIONS

THEOREM (BALOGH, DONG, LIDICKÝ, MANI, ZHAO)

If \vec{T}_k is forbidden, then  + $\left(1 + \frac{1}{\sqrt{2}}\right)$  ≤ 1 for all k .

QUESTION

If \vec{T}_k is forbidden, then  + $\left(1 - \frac{1}{k}\right)^{1-k}$  ≤ 1 for all k ?



Temporary page!

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If you rerun the document (without altering it) this surplus page will go away, because \LaTeX now knows how many pages to expect for this document.