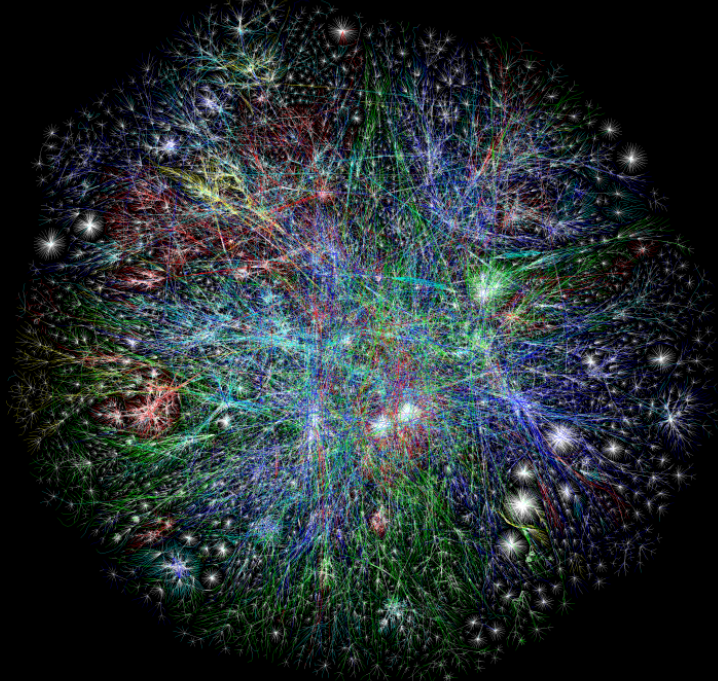


# FLAG ALGEBRAS AND ITS APPLICATION

Bernard Lidický



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Tiruchirappalli, India  
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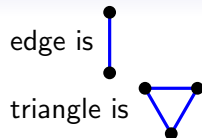


We will only consider large graphs (or networks).

Here is a graph of the internet from a while back to show there are large graphs that are interesting.

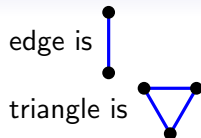
## INSPIRATIONAL PROBLEM

- Let  $n$  be a fixed number of vertices in a graph  $G$ .
- Assume  $G$  has  $m$  edges.
- What is the number of triangles in  $G$ ?



## INSPIRATIONAL PROBLEM

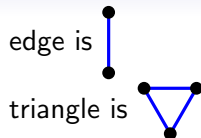
- Let  $n$  be a fixed number of vertices in a graph  $G$ .
- Assume  $G$  has  $m$  edges.  $\in [0, \binom{n}{2}]$
- What is the number of triangles in  $G$ ?  $\in [0, \binom{n}{3}]$



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Liu, Pikhurko, Staden 2020 (144 pages)



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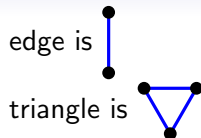
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Consider  $n \rightarrow \infty$ .

# Edges =  $p\binom{n}{2}$

# Triangles =  $t\binom{n}{3}$

Now  $p, t \in [0, 1]$ .



# INSPIRATIONAL PROBLEM

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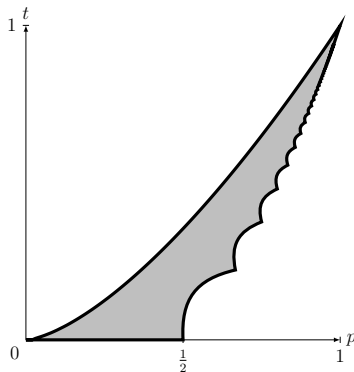
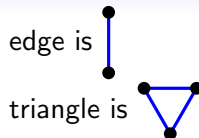
# Edges =  $p \binom{n}{2}$

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Now  $p, t \in [0, 1]$ .

Upper bound  $p^{3/2}$  Kruskal-Katona 1964

Asymptotic lower bound by Razborov 2008





## Flag Algebras and Its Application

## Inspirational Problem

## INSPIRATIONAL PROBLEM

- Let  $n$  be a fixed number of vertices in a graph  $G$ .
- Assume  $G$  has  $m$  edges.  $\in [0, \binom{n}{2}]$
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Liu, Pikhurko, Staden 2020 (144 pages)

Consider  $n \rightarrow \infty$ .  
 # Edges  $= p \binom{n}{2}$   
 # Triangles  $= t \binom{n}{3}$   
 Now  $p, t \in [0, 1]$ .

Upper bound  $p^{3/2}$  Kruskal-Katona 1964  
 Asymptotic lower bound by Razborov 2008

edge is  
triangle is




The edge (0,0) to (1/2,0) is Mantel's theorem. The sharp points are Turán graphs. The figure is exaggerated to show the idea of the shape.

# FLAG ALGEBRAS

Seminal paper:

Razborov, Flag Algebras, *Journal of Symbolic Logic* **72** (2007), 1239–1282.

David P. Robbins Prize by AMS for Razborov in 2013 over 300 citations (on Google)



## EXAMPLE

If density of edges is  $p$ , what is the minimum density of triangles?

- Designed to attack extremal problems.
- Works well if constraints as well as desired value can be computed by checking small subgraphs (or average over small subgraphs).
- The results are for the limit as graphs get very large.

# EXAMPLE EXTREMAL PROBLEM

## THEOREM (MANTEL 1907)

*Every  $n$ -vertex triangle-free graph contains at most  $\frac{1}{4}n^2$  edges.*



## PROBLEM

*Maximize a graph parameter (# of edges) over a class of graphs (triangle-free).*

- local condition and global parameter (computable locally)
- threshold
- bound and extremal example

# PROOF OF MANTEL'S THEOREM

## THEOREM (MANTEL 1907)

*In every  $n$ -vertex triangle-free graph  $|E| \leq \frac{1}{4}n^2$ .*

PROOF.

$$n|E| \geq \sum_{ij \in E} \underbrace{(d_i + d_j)}_{\leq n} = \sum_{i \in V} d_i^2 \geq \frac{(\sum_{i \in V} d_i)^2}{n} = \frac{4|E|^2}{n}$$

□

Cauchy-Schwarz  $(\sum_i a_i b_i)^2 \leq \sum_i a_i^2 \cdot \sum_i b_i^2$  with  $b_i = 1$ .

Cauchy-Schwarz  $(\sum_i a_i 1)^2 \leq \sum_i a_i^2 \cdot \sum_i 1^2$ .

## Flag Algebras and Its Application

## └ Flag Algebras

## └ Proof of Mantel's Theorem

## PROOF OF MANTEL'S THEOREM

THEOREM (MANTEL 1907)

In every  $n$ -vertex triangle-free graph  $|E| \leq \frac{1}{4}n^2$ .

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Cauchy-Schwarz  $(\sum_i a_i b_i)^2 \leq \sum_i a_i^2 \cdot \sum_i b_i^2$  with  $b_i = 1$ .Cauchy-Schwarz  $(\sum_i a_i 1)^2 \leq \sum_i a_i^2 \cdot \sum_i 1^2$ .

□

 $|E|$  is number of edges $d_i$  is a degree of a vertex  $i$ .  $d_i + d_j \leq n$  because the graph is triangle-free.

We will try to rewrite the proof using densities and this should get us familiar with flag algebras notation.

# FLAG ALGEBRAS DEFINITIONS

Let  $G$  be a graph on  $n$  vertices.



The probability that three random vertices in  $G$  span a triangle, i.e.  
 $\# \text{triangle} / \binom{n}{3}$ .

2025-12-12

# Flag Algebras and Its Application

## └ Flag Algebras

### └ Flag algebras definitions

The last click the  $= 1$  is for audience participation.

#### FLAG ALGEBRAS DEFINITIONS

Let  $G$  be a graph on  $n$  vertices.



The probability that three random vertices in  $G$  span a triangle, i.e.

$$\frac{\# \nabla}{\binom{n}{3}}.$$

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The probability that three random vertices in  $G$  span exactly two edges.  $\# \text{path} / \binom{n}{3}$ .



# Flag Algebras and Its Application

## └ Flag Algebras

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The probability that a random vertex other than  $\boxed{1}$  is adjacent to  $\boxed{1}$   $= \deg(\boxed{1}) / (n - 1)$ .

# Flag Algebras and Its Application

## └ Flag Algebras

### └ Flag algebras definitions

#### FLAG ALGEBRAS DEFINITIONS

Let  $G$  be a graph on  $n$  vertices.



The probability that three random vertices in  $G$  span a triangle, i.e.  $\# \nabla / \binom{n}{3}$ .



The probability that three random vertices in  $G$  span exactly two edges.  $\# \vee / \binom{n}{3}$ .



The probability that a random vertex other than  $x$  is adjacent to  $x$ .  $\deg(x)/(n-1)$ .

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# FLAG ALGEBRAS DEFINITIONS

Let  $G$  be a graph on  $n$  vertices.



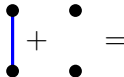
The probability that three random vertices in  $G$  span a triangle, i.e.  $\# \text{triangle} / \binom{n}{3}$ .



The probability that three random vertices in  $G$  span exactly two edges.  $\# \text{path} / \binom{n}{3}$ .



The probability that a random vertex other than  $\boxed{1}$  is adjacent to  $\boxed{1}$   $= \deg(\boxed{1}) / (n - 1)$ .



# Flag Algebras and Its Application

## └ Flag Algebras

### └ Flag algebras definitions

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Let  $G$  be a graph on  $n$  vertices.



The probability that three random vertices in  $G$  span a triangle, i.e.  $\# \nabla / \binom{n}{3}$ .



The probability that three random vertices in  $G$  span exactly two edges.  $\# \vee / \binom{n}{3}$ .



The probability that a random vertex other than  $\square$  is adjacent to  $\square$ .  $\deg(\square)/(n-1)$ .



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# FLAG ALGEBRAS DEFINITIONS

Let  $G$  be a graph on  $n$  vertices.



The probability that three random vertices in  $G$  span a triangle, i.e.  $\# \text{triangle} / \binom{n}{3}$ .



The probability that three random vertices in  $G$  span exactly two edges.  $\# \text{path} / \binom{n}{3}$ .



The probability that a random vertex other than  $\boxed{1}$  is adjacent to  $\boxed{1}$   $= \deg(\boxed{1}) / (n - 1)$ .

$$\text{edge} + \text{isolated vertex} = 1$$

# Flag Algebras and Its Application

## └ Flag Algebras

### └ Flag algebras definitions

#### FLAG ALGEBRAS DEFINITIONS

Let  $G$  be a graph on  $n$  vertices.



The probability that three random vertices in  $G$  span a triangle, i.e.  $\# \nabla / \binom{n}{3}$ .



The probability that three random vertices in  $G$  span exactly two edges.  $\# \nabla / \binom{n}{3}$ .



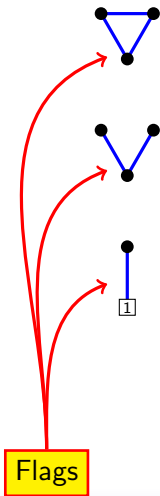
The probability that a random vertex other than  $\blacksquare$  is adjacent to  $\blacksquare$ .  $\deg(\blacksquare)/(n-1)$ .

$$\blacksquare \rightarrow \bullet = 1$$

The last click the  $= 1$  is for audience participation.

# FLAG ALGEBRAS DEFINITIONS

Let  $G$  be a graph on  $n$  vertices.



The probability that three random vertices in  $G$  span a triangle, i.e.  $\# \text{triangle} / \binom{n}{3}$ .

The probability that three random vertices in  $G$  span exactly two edges.  $\# \text{path of 2 edges} / \binom{n}{3}$ .

The probability that a random vertex other than  $\boxed{1}$  is adjacent to  $\boxed{1}$   $= \deg(\boxed{1}) / (n - 1)$ .

$$\text{edge} + \text{non-edge} = 1$$



# Flag Algebras and Its Application

## └ Flag Algebras

## └ Flag algebras definitions

### FLAG ALGEBRAS DEFINITIONS

Let  $G$  be a graph on  $n$  vertices.



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The probability that three random vertices in  $G$  span exactly two edges.  $\# \nabla / \binom{n}{3}$ .



The probability that a random vertex other than  $\blacksquare$  is adjacent to  $\blacksquare$ .  $\deg(\blacksquare)/(n-1)$ .



Flags

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## THEOREM (MANTEL 1907)

In every  $n$ -vertex triangle-free graph  $|E| \leq \frac{1}{4}n^2$ .

$$n^2|E| \geq n \sum_{ij \in E} \underbrace{(d_i + d_j)}_{\leq n} = \sum_{i \in V} 1^2 \cdot \sum_{i \in V} d_i^2 \geq \left( \sum_{i \in V} 1 \cdot d_i \right)^2 = 4|E|^2$$

$$|E| = \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \binom{n}{2} \approx \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \frac{n^2}{2}, \quad d_1 = \begin{array}{c} \bullet \\ | \\ \boxed{1} \end{array} (n-1) \approx \begin{array}{c} \bullet \\ | \\ \boxed{1} \end{array} \cdot n, \quad 1 = \boxed{1}$$

# THEOREM (MANTEL 1907)

In every  $n$ -vertex triangle-free graph  $|E| \leq \frac{1}{4}n^2$ . If  = 0 then   $\leq \frac{1}{2}$

$$n^2|E| \geq n \sum_{ij \in E} \underbrace{(d_i + d_j)}_{\leq n} = \sum_{i \in V} 1^2 \cdot \sum_{i \in V} d_i^2 \geq \left( \sum_{i \in V} 1 \cdot d_i \right)^2 = 4|E|^2$$

$$|E| = \text{img edge} \binom{n}{2} \approx \text{img edge} \frac{n^2}{2}, \quad d_1 = \text{img vertex 1} (n-1) \approx \text{img vertex 1} \cdot n, \quad 1 = \boxed{1}$$

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$$n^2 \times \text{edge} \times \frac{n^2}{2} \geq \sum_{\mathbb{1} \in V} \mathbb{1}^2 \cdot \sum_{\mathbb{1} \in V} \left( \text{edge} \frac{n}{\mathbb{1}} \right)^2 \geq \left( \sum_{\mathbb{1} \in V} \mathbb{1} \cdot \text{edge} \frac{n}{\mathbb{1}} \right)^2 = 4 \left( \text{edge} \frac{n^2}{2} \right)^2$$

$$|E| = \text{edge} \binom{n}{2} \approx \text{edge} \frac{n^2}{2}, \quad d_1 = \text{edge} \frac{n-1}{\mathbb{1}} \approx \text{edge} \frac{n}{\mathbb{1}} \cdot n, \quad 1 = \mathbb{1}$$

# THEOREM (MANTEL 1907)

In every  $n$ -vertex triangle-free graph  $|E| \leq \frac{1}{4}n^2$ . If  = 0 then   $\leq \frac{1}{2}$

$$n^2|E| \geq n \sum_{ij \in E} \underbrace{(d_i + d_j)}_{\leq n} = \sum_{i \in V} 1^2 \cdot \sum_{i \in V} d_i^2 \geq \left( \sum_{i \in V} 1 \cdot d_i \right)^2 = 4|E|^2$$

$$n^2 \times \text{edge} \frac{n^2}{2} \geq \sum_{\boxed{1} \in V} \boxed{1}^2 \cdot \sum_{\boxed{1} \in V} \left( \text{edge} \frac{n}{\boxed{1}} \right)^2 \geq \left( \sum_{\boxed{1} \in V} \boxed{1} \cdot \text{edge} \frac{n}{\boxed{1}} \right)^2 = 4 \left( \text{edge} \frac{n^2}{2} \right)^2$$

$$\frac{1}{2} \text{edge} \geq \frac{1}{n} \sum_{\boxed{1} \in V} \boxed{1}^2 \cdot \frac{1}{n} \sum_{\boxed{1} \in V} \left( \text{edge} \frac{n}{\boxed{1}} \right)^2 \geq \left( \frac{1}{n} \sum_{\boxed{1} \in V} \boxed{1} \cdot \text{edge} \frac{n}{\boxed{1}} \right)^2 = \text{edge}^2$$

$$|E| = \text{edge} \binom{n}{2} \approx \text{edge} \frac{n^2}{2}, \quad d_1 = \text{edge} \frac{n-1}{\boxed{1}} \approx \text{edge} \cdot n, \quad 1 = \boxed{1}$$

## Flag Algebras and Its Application

## └ Flag Algebras

THEOREM (MANTTEL 1907)

In every  $n$ -vertex triangle-free graph  $|E| \leq \frac{1}{2}n^2$ . If  $\nabla = 0$  then  $|E| \leq \frac{1}{2}n^2$ 

$$n^2|E| \geq n \sum_{\substack{d_i + d_j \\ \leq n}} (d_i + d_j) = \sum_{i \in V} 1^2 \cdot \sum_{j \in V} d_i^2 \geq \left( \sum_{i \in V} 1 \cdot d_i \right)^2 = 4|E|^2$$

$$n^2 \times \frac{n^2}{2} \geq \sum_{\square \in V} \square^2 \cdot \sum_{\square \in V} \left( \frac{n}{\square} \right)^2 \geq \left( \sum_{\square \in V} \square \cdot \frac{n}{\square} \right)^2 = 4 \left( \frac{n^2}{2} \right)^2$$

$$\frac{1}{2} \geq \frac{1}{n} \sum_{\square \in V} \square^2 \cdot \frac{1}{n} \sum_{\square \in V} \left( \frac{n}{\square} \right)^2 \geq \left( \frac{1}{n} \sum_{\square \in V} \square \cdot \frac{n}{\square} \right)^2 = 4$$

$$|E| = \frac{n^2}{2} \approx \frac{n^2}{2}, \quad d_i = \frac{n}{\square} (n-1) \approx \frac{n}{\square} \cdot n, \quad 1 = \square$$

We are ignoring lower order terms and approximate  $\binom{n}{k}$  by  $\frac{n^k}{k!}$ .

$$n^2|E| \geq n \sum_{ij \in E} \underbrace{(d_i + d_j)}_{\leq n} = \sum_{i \in V} 1^2 \cdot \sum_{i \in V} d_i^2 \geq \left( \sum_{i \in V} 1 \cdot d_i \right)^2 = 4|E|^2$$

$$\frac{1}{2} \text{---} \bullet \geq \frac{1}{n} \sum_{\mathbb{1} \in V} \mathbb{1}^2 \cdot \frac{1}{n} \sum_{\mathbb{1} \in V} \left( \text{---} \bullet_{\mathbb{1}} \right)^2 \geq \left( \frac{1}{n} \sum_{\mathbb{1} \in V} \mathbb{1} \cdot \text{---} \bullet_{\mathbb{1}} \right)^2 = \text{---} \bullet^2$$

$$\frac{1}{n} \sum_{\mathbb{1} \in V} f = \llbracket f \rrbracket$$

Cauchy-Schwarz:  $\llbracket f^2 \rrbracket \cdot \llbracket g^2 \rrbracket \geq \llbracket f \cdot g \rrbracket^2$ . In particular,  $\llbracket f^2 \rrbracket \geq 0$ .

$$n^2|E| \geq n \sum_{ij \in E} \underbrace{(d_i + d_j)}_{\leq n} = \sum_{i \in V} 1^2 \cdot \sum_{i \in V} d_i^2 \geq \left( \sum_{i \in V} 1 \cdot d_i \right)^2 = 4|E|^2$$

$$\frac{1}{2} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \geq \frac{1}{n} \sum_{\mathbb{1} \in V} \mathbb{1}^2 \cdot \frac{1}{n} \sum_{\mathbb{1} \in V} \left( \begin{array}{c} \bullet \\ | \\ \mathbb{1} \end{array} \right)^2 \geq \left( \frac{1}{n} \sum_{\mathbb{1} \in V} \mathbb{1} \cdot \begin{array}{c} \bullet \\ | \\ \mathbb{1} \end{array} \right)^2 = \begin{array}{c} \bullet^2 \\ | \\ \bullet \end{array}$$

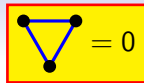
$$\frac{1}{2} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \geq \left[ \mathbb{1}^2 \right] \cdot \left[ \begin{array}{c} \bullet^2 \\ | \\ \mathbb{1} \end{array} \right] \geq \left[ \mathbb{1} \cdot \begin{array}{c} \bullet \\ | \\ \mathbb{1} \end{array} \right]^2 = \begin{array}{c} \bullet^2 \\ | \\ \bullet \end{array}$$

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
$$\frac{1}{2} \geq \left[ \begin{array}{c} \boxed{1}^2 \end{array} \right] \cdot \left[ \begin{array}{c} \bullet^2 \\ \boxed{1} \end{array} \right]$$



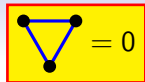
$$= 0$$

$\boxed{1}$  = not choosing anything = 1

 = probability of choosing a vertex ...  $\deg(\boxed{1})/(n-1)$


 = probability of choosing two distinct vertices ...  $\binom{\deg(\boxed{1})}{2} / \binom{n-1}{2}$


$$\frac{1}{2} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \geq \left[ \begin{array}{c} \boxed{1}^2 \end{array} \right] \cdot \left[ \begin{array}{c} \bullet^2 \\ | \\ \boxed{1} \end{array} \right]$$



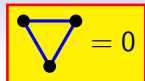
$$\begin{array}{c} \bullet \\ | \\ \boxed{1} \end{array} \times \begin{array}{c} \bullet \\ | \\ \boxed{1} \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \text{---} \\ | \\ \boxed{1} \end{array} + o(1) = \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ | \\ \boxed{1} \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ | \\ \boxed{1} \end{array} + o(1)$$

$\boxed{1}$  = not choosing anything = 1

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 = probability of choosing two distinct vertices ...  $\binom{\deg(\boxed{1})}{2} / \binom{n-1}{2}$


$$\frac{1}{2} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \geq \left[ \begin{array}{c} \boxed{1} \\ | \\ \boxed{1} \end{array} \right]^2 \cdot \left[ \begin{array}{c} \bullet^2 \\ | \\ \boxed{1} \end{array} \right] = 1 \cdot \left[ \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \boxed{1} \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \boxed{1} \end{array} \right] = \left[ \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \boxed{1} \end{array} \right]$$



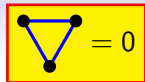
$$\begin{array}{c} \bullet \\ | \\ \boxed{1} \end{array} \times \begin{array}{c} \bullet \\ | \\ \boxed{1} \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \boxed{1} \end{array} + o(1) = \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \boxed{1} \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \boxed{1} \end{array} + o(1)$$

$\boxed{1}$  = not choosing anything = 1

 = probability of choosing a vertex ...  $\deg(\boxed{1})/(n-1)$

 = probability of choosing two distinct vertices ...  $\binom{\deg(\boxed{1})}{2} / \binom{n-1}{2}$


$$\frac{1}{2} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \geq \left[ \begin{array}{c} \square \\ | \\ \square \end{array} \right]^2 \cdot \left[ \begin{array}{c} \bullet^2 \\ | \\ \square \end{array} \right] = 1 \cdot \left[ \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \square \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \square \end{array} \right] = \left[ \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \square \end{array} \right]$$




$$\begin{array}{c} \bullet \\ | \\ \square \end{array} \times \begin{array}{c} \bullet \\ | \\ \square \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \square \end{array} + o(1) = \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \square \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \square \end{array} + o(1)$$

$$\begin{array}{c} \bullet \\ | \\ \square \end{array} \times \begin{array}{c} \bullet \\ | \\ \square \end{array} = \frac{1}{2} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \square \end{array} = \frac{1}{2} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \square \end{array} + \frac{1}{2} \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \square \end{array}$$

$\square$  = not choosing anything = 1

 = probability of choosing a vertex ...  $\deg(\square)/(n-1)$

 = probability of choosing two distinct vertices ...  $\binom{\deg(\square)}{2} / \binom{n-1}{2}$

## Flag Algebras and Its Application

## └ Flag Algebras

$$\frac{1}{2} \left[ \begin{array}{c} \text{---} \\ | \\ \square \end{array} \right] \cdot \left[ \begin{array}{c} \text{---} \\ | \\ \square \end{array} \right] = 1 \cdot \left[ \begin{array}{c} \text{---} \\ | \\ \square \end{array} \right] + \left[ \begin{array}{c} \text{---} \\ | \\ \square \end{array} \right] = \left[ \begin{array}{c} \text{---} \\ | \\ \square \end{array} \right]$$

$$\left[ \begin{array}{c} \text{---} \\ | \\ \square \end{array} \right] \times \left[ \begin{array}{c} \text{---} \\ | \\ \square \end{array} \right] = \left[ \begin{array}{c} \text{---} \\ | \\ \square \end{array} \right] + o(1) = \left[ \begin{array}{c} \text{---} \\ | \\ \square \end{array} \right] + o(1)$$

$$\left[ \begin{array}{c} \text{---} \\ | \\ \square \end{array} \right] \times \left[ \begin{array}{c} \text{---} \\ | \\ \square \end{array} \right] = \frac{1}{2} \left[ \begin{array}{c} \text{---} \\ | \\ \square \end{array} \right] + \frac{1}{2} \left[ \begin{array}{c} \text{---} \\ | \\ \square \end{array} \right] + \frac{1}{2} \left[ \begin{array}{c} \text{---} \\ | \\ \square \end{array} \right]$$

$\square$  = not choosing anything = 1

$\left[ \begin{array}{c} \text{---} \\ | \\ \square \end{array} \right]$  = probability of choosing a vertex ...  $\deg(\square)/(n-1)$

$\left[ \begin{array}{c} \text{---} \\ | \\ \square \end{array} \right] \times \left[ \begin{array}{c} \text{---} \\ | \\ \square \end{array} \right]$  = probability of choosing two distinct vertices ...  $\binom{\deg(\square)}{2}/\binom{n-1}{2}$



Probability of choosing a vertex inducing an edge with the fixed vertex 1.

Probability of choosing a pair of distinct vertices each being in an edge with the fixed vertex 1.

Notice on the left each pair of vertices counted twice!

We will ignore  $o(1)$  in the future

$$\frac{1}{2} \geq \left[ \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ \boxed{1} \end{array} \right]$$

$$\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \quad \boxed{1} = \text{probability of choosing two distinct vertices} \dots \binom{\deg(\boxed{1})}{2} / \binom{n-1}{2}$$

$$\sum_{\boxed{1} \in V} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \quad \boxed{1} \binom{n-1}{2} = \# \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array}$$

$$\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} = \text{probability of choosing three distinct vertices} \dots \# \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} / \binom{n}{3}$$



$$\frac{1}{2} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \geq \left[ \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \boxed{1} \end{array} \right] = \frac{1}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$$

$$\left[ \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \boxed{1} \end{array} \right] = \frac{1}{n} \sum_{\boxed{1} \in V} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \boxed{1} \end{array} = \frac{1}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$$

$$\begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \boxed{1} \end{array} = \text{probability of choosing two distinct vertices} \dots \binom{\deg(\boxed{1})}{2} / \binom{n-1}{2}$$

$$\sum_{\boxed{1} \in V} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \boxed{1} \end{array} \binom{n-1}{2} = \# \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$$

$$\begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \text{probability of choosing three distinct vertices} \dots \# \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} / \binom{n}{3}$$



$$\frac{1}{2} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \geq \left[ \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \boxed{1} \end{array} \right] = \frac{1}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$$

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \geq \frac{2}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$$

$$\left[ \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \boxed{1} \end{array} \right] = \frac{1}{n} \sum_{\boxed{1} \in V} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \boxed{1} \end{array} = \frac{1}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$$

$$\begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \boxed{1} \end{array} = \text{probability of choosing two distinct vertices} \dots \binom{\deg(\boxed{1})}{2} / \binom{n-1}{2}$$


$$\sum_{\boxed{1} \in V} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \boxed{1} \end{array} \binom{n-1}{2} = \# \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$$


$$\begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \text{probability of choosing three distinct vertices} \dots \# \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} / \binom{n}{3}$$

$$\text{edge} \geq \frac{2}{3} \text{V}$$

$$|E| = \frac{1}{n} \left( \sum_{\text{edge}} 1 + \sum_{\text{V}} 2 + \sum_{\text{triangle}} 3 \right)$$

$$\text{edge} \binom{n}{2} \approx \frac{1}{n} \text{edge} \binom{n}{3} + \frac{2}{n} \text{V} \binom{n}{3} + \frac{3}{n} \text{triangle} \binom{n}{3}$$

 = probability of choosing an edge ...  $|E|/\binom{n}{2}$


 = probability of choosing an triple ...  $\# \text{triple} / \binom{n}{2}$


$$\text{edge} \geq \frac{2}{3} \text{V}$$

$$\text{edge} = \frac{1}{3} \text{H} + \frac{2}{3} \text{V} + \frac{3}{3} \text{triangle}$$




$$|E| = \frac{1}{n} \left( \sum_{\text{H}} 1 + \sum_{\text{V}} 2 + \sum_{\text{triangle}} 3 \right)$$

$$\text{edge} \binom{n}{2} \approx \frac{1}{n} \text{H} \binom{n}{3} + \frac{2}{n} \text{V} \binom{n}{3} + \frac{3}{n} \text{triangle} \binom{n}{3}$$










 = probability of choosing an edge ...  $|E|/\binom{n}{2}$





 = probability of choosing a triple ...  $\# \text{triples} / \binom{n}{3}$

# PROOF RECAP





If  =  = 0 then   $\leq 1/2$ .

$$n^2|E| \geq \sum_{i \in V} 1^2 \cdot \sum_{i \in V} d_i^2 \geq \left( \sum_{i \in V} 1 \cdot d_i \right)^2 = 4|E|^2$$

$$\frac{1}{2} \text{  } \geq \frac{1}{3} \text{  } = \left[ \text{  } \right] = \left[ \text{  }^2 \right] \cdot \left[ \text{  }^2 \right] \geq \left[ \text{  } \times \text{  } \right]^2 = \left[ \text{  } \right]^2 = \text{  }$$

$$\text{  } = \frac{1}{3} \text{  } + \frac{2}{3} \text{  } + \frac{3}{3} \text{  }$$

$$\left[ \text{  } \right] = \frac{1}{n} \sum_{\text{  } \in V} \text{  } = \frac{1}{3} \text{  }$$

$$\text{  } \times \text{  } = \text{  } + \text{  }$$

$$\left[ f^2 \right] \cdot \left[ g^2 \right] \geq \left[ f \cdot g \right]^2$$

DIFFERENT PROOF OF  $\triangle = 0$  IMPLIES  $\text{I} \leq 1/2$

$$\text{I} = \frac{1}{3} \begin{array}{c} \bullet \\ \text{---} \bullet \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{3}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$$

DIFFERENT PROOF OF  $\triangle = 0$  IMPLIES  $\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \leq 1/2$

$$\begin{aligned}
 \begin{array}{c} \bullet \\ | \\ \bullet \end{array} &= \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{3}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \\
 &\leq \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{1}{2} \left[ \left( \begin{array}{c} \bullet \\ | \\ \boxed{1} \end{array} - \begin{array}{c} \bullet \\ \boxed{1} \end{array} \right)^2 \right]
 \end{aligned}$$

DIFFERENT PROOF OF  $\triangle = 0$  IMPLIES  $\left| \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right| \leq 1/2$

$$\left| \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right| = \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{3}{3} \triangle$$

$$\leq \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{1}{2} \left[ \left( \begin{array}{c} \bullet \\ | \\ \boxed{1} \end{array} - \begin{array}{c} \bullet \\ | \\ \boxed{1} \end{array} \right)^2 \right]$$

$$= \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{1}{2} \left[ \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \boxed{1} \end{array} + \triangle - \begin{array}{c} \bullet \\ | \\ \boxed{1} \end{array} - \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \boxed{1} \end{array} + \begin{array}{c} \bullet \\ | \\ \boxed{1} \end{array} + \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \boxed{1} \end{array} \right]$$

DIFFERENT PROOF OF  $\triangle = 0$  IMPLIES  $\left| \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right| \leq 1/2$

$$\left| \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right| = \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{3}{3} \triangle$$

$$\leq \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{1}{2} \left[ \left( \begin{array}{c} \bullet \\ | \\ \boxed{1} \end{array} - \begin{array}{c} \bullet \\ | \\ \boxed{1} \end{array} \right)^2 \right]$$

$$= \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{1}{2} \left[ \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \boxed{1} \end{array} + \triangle - \begin{array}{c} \bullet \\ | \\ \boxed{1} \end{array} - \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \boxed{1} \end{array} + \begin{array}{c} \bullet \\ | \\ \boxed{1} \end{array} + \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \boxed{1} \end{array} \right]$$

$$= \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{1}{2} \left( \frac{1}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} - \frac{2}{3} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} - \frac{2}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} \right)$$



DIFFERENT PROOF OF  $\triangle = 0$  IMPLIES  $\left| \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right| \leq 1/2$

$$\begin{aligned}
 \left| \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right| &= \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{3}{3} \triangle \\
 &\leq \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{1}{2} \left[ \left( \begin{array}{c} \bullet \\ | \\ \boxed{1} \end{array} - \begin{array}{c} \bullet \\ | \\ \boxed{1} \end{array} \right)^2 \right] \\
 &= \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{1}{2} \left[ \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \boxed{1} \end{array} + \triangle - \begin{array}{c} \bullet \\ | \\ \boxed{1} \end{array} - \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \boxed{1} \end{array} + \begin{array}{c} \bullet \\ | \\ \boxed{1} \end{array} + \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \boxed{1} \end{array} \right] \\
 &= \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{1}{2} \left( \frac{1}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} - \frac{2}{3} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} - \frac{2}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} \right) \\
 &= \frac{1}{2} \begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ \bullet \end{array} + \frac{1}{6} \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} + \frac{1}{2} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}
 \end{aligned}$$

DIFFERENT PROOF OF  $\triangle = 0$  IMPLIES  $\left| \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right| \leq 1/2$

$$\begin{aligned}
 \left| \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right| &= \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{3}{3} \triangle \\
 &\leq \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{1}{2} \left[ \left( \begin{array}{c} \bullet \\ | \\ \boxed{1} \end{array} - \begin{array}{c} \bullet \\ | \\ \boxed{1} \end{array} \right)^2 \right] \\
 &= \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{1}{2} \left[ \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \boxed{1} \end{array} + \triangle - \begin{array}{c} \bullet \\ | \\ \boxed{1} \end{array} - \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \boxed{1} \end{array} + \begin{array}{c} \bullet \\ | \\ \boxed{1} \end{array} + \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \boxed{1} \end{array} \right] \\
 &= \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{1}{2} \left( \frac{1}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} - \frac{2}{3} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} - \frac{2}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} \right) \\
 &= \frac{1}{2} \begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ \bullet \end{array} + \frac{1}{6} \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} + \frac{1}{2} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \\
 &\leq \frac{1}{2} \underbrace{\left( \begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ \bullet \end{array} + \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right)}_{=1} = \frac{1}{2}
 \end{aligned}$$

# AUTOMATED APPROACH

$$\begin{aligned}
 \text{flag} &\leq \text{flag} + \left[ \left( \text{flag} - \frac{1}{2} \text{sum of squares} \right)^2 \right] = \frac{1}{2} \text{sum of squares} + \frac{1}{6} \text{flag} + \frac{1}{2} \text{flag} \\
 &\leq \frac{1}{2} \left( \text{sum of squares} + \text{flag} + \text{flag} \right) = \frac{1}{2}
 \end{aligned}$$

In general as sum of squares

$$f \leq f + \sum_h \llbracket h^2 \rrbracket = \sum_{G \in \mathcal{F}_n} c_G \cdot G \leq \max_{G \in \mathcal{F}_n} c_G$$

$f, g$  linear combination of flags

$\mathcal{F}_n$  ... flags on  $n$  vertices

SOS proofs can be optimized by semidefinite programming

# SUM OF SQUARES

$$\begin{array}{|c|c|} \bullet & \bullet \\ \hline \bullet & \bullet \end{array} \leq \begin{array}{|c|c|} \bullet & \bullet \\ \hline \boxed{1} & \boxed{1} \end{array} + \left[ \left( \begin{array}{|c|c|} \bullet & \bullet \\ \hline \boxed{1} & \boxed{1} \end{array} - \begin{array}{|c|c|} \bullet & \bullet \\ \hline \bullet & \bullet \end{array} \right)^2 \right] = \frac{1}{2} \begin{array}{|c|c|} \bullet & \bullet \\ \hline \bullet & \bullet \end{array} + \frac{1}{6} \begin{array}{|c|c|} \bullet & \bullet \\ \hline \bullet & \bullet \end{array} + \frac{1}{2} \begin{array}{|c|c|} \bullet & \bullet \\ \hline \bullet & \bullet \end{array} \leq \frac{1}{2}$$

In general as sum of squares

$$f \leq f + \sum_h \llbracket h^2 \rrbracket = \sum_{G \in \mathcal{F}_n} c_G \cdot G \leq \max_{G \in \mathcal{F}_n} c_G$$

Semidefinite matrix

$$\begin{array}{|c|c|} \bullet & \bullet \\ \hline \bullet & \bullet \end{array} \leq \begin{array}{|c|c|} \bullet & \bullet \\ \hline \boxed{1} & \boxed{1} \end{array} + \left[ \begin{array}{|c|c|} \bullet & \bullet \\ \hline \boxed{1} & \boxed{1} \end{array} \underbrace{\begin{pmatrix} a & c \\ c & b \end{pmatrix}}_{=M \succeq 0} \begin{array}{|c|c|} \bullet & \bullet \\ \hline \boxed{1} & \boxed{1} \end{array}^T \right] = \sum_{G \in \mathcal{F}_n} c_{G,M} \cdot G \leq \max_{G \in \mathcal{F}_n} c_{G,M}$$

# SUM OF SQUARES

$$\begin{array}{|c|c|} \bullet & \bullet \\ \hline \bullet & \bullet \end{array} \leq \begin{array}{|c|c|} \bullet & \bullet \\ \hline \boxed{1} & \boxed{1} \end{array} + \left[ \left( \begin{array}{|c|c|} \bullet & \bullet \\ \hline \boxed{1} & \boxed{1} \end{array} - \begin{array}{|c|c|} \bullet & \bullet \\ \hline \bullet & \bullet \end{array} \right)^2 \right] = \frac{1}{2} \begin{array}{|c|c|} \bullet & \bullet \\ \hline \bullet & \bullet \end{array} + \frac{1}{6} \begin{array}{|c|c|} \bullet & \bullet \\ \hline \bullet & \bullet \end{array} + \frac{1}{2} \begin{array}{|c|c|} \bullet & \bullet \\ \hline \bullet & \bullet \end{array} \leq \frac{1}{2}$$

In general as sum of squares

$$f \leq f + \sum_h \llbracket h^2 \rrbracket = \sum_{G \in \mathcal{F}_n} c_G \cdot G \leq \max_{G \in \mathcal{F}_n} c_G$$

Semidefinite matrix

$$\begin{array}{|c|c|} \bullet & \bullet \\ \hline \bullet & \bullet \end{array} \leq \begin{array}{|c|c|} \left( \begin{array}{|c|c|} \bullet & \bullet \\ \hline \boxed{1} & \boxed{1} \end{array}, \begin{array}{|c|c|} \bullet & \bullet \\ \hline \boxed{1} & \boxed{1} \end{array} \right) \underbrace{\begin{pmatrix} a & c \\ c & b \end{pmatrix}}_{=M \succeq 0} \left( \begin{array}{|c|c|} \bullet & \bullet \\ \hline \boxed{1} & \boxed{1} \end{array}, \begin{array}{|c|c|} \bullet & \bullet \\ \hline \boxed{1} & \boxed{1} \end{array} \right)^T \end{array} = \sum_{G \in \mathcal{F}_n} c_{G,M} \cdot G \leq \min_{M \succeq 0} \max_{G \in \mathcal{F}_n} c_{G,M}$$

## Flag Algebras and Its Application

└ Flag Algebras

└ Sum of squares

Make the subscript of Max under

$$\mathbb{I} \leq \mathbb{I} + \left[ \begin{pmatrix} \mathbb{I} & \bullet \\ \bullet & -\mathbb{I} \end{pmatrix}^2 \right] = \frac{1}{2} \bullet \bullet + \frac{1}{6} \bullet \bullet + \frac{1}{2} \mathbb{V} \leq \frac{1}{2}$$

In general as sum of squares

$$f \leq f + \sum_n [h_n^2] = \sum_{G \in \mathcal{F}_n} c_G \cdot G \leq \max_{G \in \mathcal{F}_n} c_G$$

Semidefinite matrix

$$\mathbb{I} \leq \mathbb{I} + \left[ \begin{pmatrix} \mathbb{I} & \bullet \\ \bullet & -\mathbb{I} \end{pmatrix} \underbrace{\begin{pmatrix} a & c \\ c & b \end{pmatrix}}_{=M \in \mathbb{S}} \begin{pmatrix} \mathbb{I} & \bullet \\ \bullet & -\mathbb{I} \end{pmatrix}^T \right] = \sum_{G \in \mathcal{F}_n} c_{G,M} \cdot G \leq \min_{M \in \mathbb{S}} \max_{G \in \mathcal{F}_n} c_{G,M}$$

## **Rainbow Triangles**

2025-12-12

Flag Algebras and Its Application

└ Flag Algebras

Rainbow Triangles

<https://arxiv.org/abs/2511.21061>



# JOINTS

Joint in  $\mathbb{R}^d$  is a point where  $d$  lines that span  $\mathbb{R}^d$  intersect.

What is the maximum number of joints for  $N$  lines?

THEOREM (CHAO AND HANS YU 2023+)

*Number of joints is maximized by  $k$  hyperplanes whose intersection give  $N = \binom{k}{d-1}$  lines and  $\binom{k}{d}$  joints.*

Asymptotically by Hans Yu and Zhao 2023.

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Multijoint problem: In  $\mathbb{R}^3$ , lines of three colors, maximize rainbow joints.

hyperplane  $\rightarrow$  vertex

intersection of hyperplanes  $\rightarrow$  edge

joint  $\rightarrow$  rainbow triangle

**THEOREM (CHAO AND HANS YU 2024+)**

*In 3-edge colored graph  $\left(\# \begin{array}{c} \bullet \text{---} \bullet \\ \text{red} \\ \bullet \text{---} \bullet \\ \text{green} \\ \bullet \text{---} \bullet \\ \text{blue} \end{array}\right)^2 \leq 2 \left(\# \begin{array}{c} \bullet \text{---} \bullet \\ \text{blue} \\ \bullet \text{---} \bullet \\ \text{blue} \end{array}\right) \cdot \left(\# \begin{array}{c} \bullet \text{---} \bullet \\ \text{green} \\ \bullet \text{---} \bullet \\ \text{green} \end{array}\right) \cdot \left(\# \begin{array}{c} \bullet \text{---} \bullet \\ \text{red} \\ \bullet \text{---} \bullet \\ \text{red} \end{array}\right).$*

# JOINTS

Multijoint problem: In  $\mathbb{R}^3$ , lines of three colors, maximize rainbow joints.

hyperplane  $\rightarrow$  vertex

intersection of hyperplanes  $\rightarrow$  edge

joint  $\rightarrow$  rainbow triangle

THEOREM (CHAO AND HANS YU 2024+)

In 3-edge colored graph  $\left( \# \begin{array}{c} \bullet \text{---} \bullet \\ \text{---} \bullet \\ \bullet \end{array} \right)^2 \leq 2 \left( \# \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \right) \cdot \left( \# \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \right) \cdot \left( \# \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \right).$

In flag algebras

$$\left( \begin{array}{c} \bullet \text{---} \bullet \\ \text{---} \bullet \\ \bullet \end{array} \right)^2 \leq 9 \cdot \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \cdot \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \cdot \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array}$$

Automated sum-of-squares proof needs 540GB RAM

# Flag Algebras and Its Application

## └ Flag Algebras

## └ Joints

Motivation is for joints - one take lined colored red, green and blue and asks for rainbow lines.

Their proof uses entropy method.

1,601,952 configurations and 540GB ram

### JOINTS

Multijoint problem: In  $\mathbb{R}^3$ , lines of three colors, maximize rainbow joints.  
 hyperplane  $\rightarrow$  vertex  
 intersection of hyperplanes  $\rightarrow$  edge  
 joint  $\rightarrow$  rainbow triangle

THEOREM (CHAO AND HANS YU 2024+)

In 3-edge colored graph  $(\# \text{rainbow triangles})^2 \leq 2 \left( \# \text{red edges} \right) \cdot \left( \# \text{green edges} \right) \cdot \left( \# \text{blue edges} \right)$ .

In flag algebras

$$(\text{rainbow triangles})^2 \leq 0$$

Automated sum-of-squares proof needs 540GB RAM

$$\begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \times \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \geq \frac{1}{3} \cdot \left( \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \right) = 4 \cdot \left[ \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \boxed{1} \quad \boxed{2} \end{array} \right]^2.$$

$$\begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \times \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \geq \frac{1}{3} \cdot \left( \begin{array}{c} \bullet \\ \diagup \text{ (blue)} \diagdown \text{ (green)} \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagup \text{ (blue)} \diagdown \text{ (blue)} \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagup \text{ (green)} \diagdown \text{ (green)} \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagup \text{ (blue)} \diagdown \text{ (red)} \\ \bullet \end{array} \right) = 4 \cdot \left[ \begin{array}{c} \bullet \\ \diagup \text{ (blue)} \diagdown \text{ (red)} \\ \boxed{1} \end{array} \right]^2.$$

$$\begin{aligned} \begin{array}{c} \bullet \\ \diagup \text{ (blue)} \diagdown \text{ (red)} \\ \bullet \end{array} &= 6 \cdot \left[ \begin{array}{c} \bullet \\ \diagup \text{ (blue)} \diagdown \text{ (red)} \\ \boxed{1} \end{array} \right] = 6 \cdot \left[ \begin{array}{c} \bullet \\ \diagup \text{ (blue)} \diagdown \text{ (red)} \\ \boxed{1} \end{array} \times \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \right] \\ &\leq 6 \cdot \sqrt{\left[ \left( \begin{array}{c} \bullet \\ \diagup \text{ (blue)} \diagdown \text{ (red)} \\ \boxed{1} \end{array} \right)^2 \right]} \cdot \sqrt{\left[ \left( \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \right)^2 \right]} \\ &= 6 \cdot \sqrt{\frac{1}{12} \left( \begin{array}{c} \bullet \\ \diagup \text{ (blue)} \diagdown \text{ (green)} \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagup \text{ (blue)} \diagdown \text{ (blue)} \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagup \text{ (green)} \diagdown \text{ (green)} \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagup \text{ (blue)} \diagdown \text{ (red)} \\ \bullet \end{array} \right)} \cdot \sqrt{\begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array}} \\ &\leq 3 \cdot \sqrt{\begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \times \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array}} \cdot \sqrt{\begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array}} = 3 \cdot \sqrt{\begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \times \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \times \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array}}. \end{aligned}$$

## Flag Algebras and Its Application


└ Flag Algebras

└ Balogh, Bradshaw, Garcia, L. 2025+


$$\begin{aligned}
 \left[ \begin{smallmatrix} \text{red} & \text{blue} \\ \text{green} & \text{blue} \end{smallmatrix} \right] &\geq \frac{1}{3} \cdot \left( \left[ \begin{smallmatrix} \text{red} & \text{blue} \\ \text{blue} & \text{blue} \end{smallmatrix} \right] + \left[ \begin{smallmatrix} \text{blue} & \text{blue} \\ \text{green} & \text{blue} \end{smallmatrix} \right] + \left[ \begin{smallmatrix} \text{blue} & \text{blue} \\ \text{blue} & \text{blue} \end{smallmatrix} \right] \right) = 4 \cdot \left[ \begin{smallmatrix} \text{blue} & \text{blue} \\ \text{blue} & \text{blue} \end{smallmatrix} \right]^2 \\
 \left[ \begin{smallmatrix} \text{red} & \text{blue} \\ \text{green} & \text{blue} \end{smallmatrix} \right] &= 6 \cdot \left[ \begin{smallmatrix} \text{blue} & \text{blue} \\ \text{blue} & \text{blue} \end{smallmatrix} \right]^2 = 6 \cdot \left[ \begin{smallmatrix} \text{blue} & \text{blue} \\ \text{blue} & \text{blue} \end{smallmatrix} \right] \times \left[ \begin{smallmatrix} \text{blue} & \text{blue} \\ \text{blue} & \text{blue} \end{smallmatrix} \right] \\
 &\leq 6 \cdot \sqrt{\left[ \begin{smallmatrix} \text{blue} & \text{blue} \\ \text{blue} & \text{blue} \end{smallmatrix} \right]^2} \cdot \sqrt{\left[ \begin{smallmatrix} \text{blue} & \text{blue} \\ \text{blue} & \text{blue} \end{smallmatrix} \right]^2} \\
 &= 6 \cdot \sqrt{\frac{1}{12} \left( \left[ \begin{smallmatrix} \text{red} & \text{blue} \\ \text{blue} & \text{blue} \end{smallmatrix} \right] + \left[ \begin{smallmatrix} \text{blue} & \text{blue} \\ \text{green} & \text{blue} \end{smallmatrix} \right] + \left[ \begin{smallmatrix} \text{blue} & \text{blue} \\ \text{blue} & \text{blue} \end{smallmatrix} \right] \right)} \cdot \sqrt{\frac{1}{12} \left( \left[ \begin{smallmatrix} \text{red} & \text{blue} \\ \text{blue} & \text{blue} \end{smallmatrix} \right] + \left[ \begin{smallmatrix} \text{blue} & \text{blue} \\ \text{green} & \text{blue} \end{smallmatrix} \right] + \left[ \begin{smallmatrix} \text{blue} & \text{blue} \\ \text{blue} & \text{blue} \end{smallmatrix} \right] \right)} \\
 &\leq 3 \cdot \sqrt{\frac{1}{12} \left( \left[ \begin{smallmatrix} \text{red} & \text{blue} \\ \text{blue} & \text{blue} \end{smallmatrix} \right] + \left[ \begin{smallmatrix} \text{blue} & \text{blue} \\ \text{green} & \text{blue} \end{smallmatrix} \right] + \left[ \begin{smallmatrix} \text{blue} & \text{blue} \\ \text{blue} & \text{blue} \end{smallmatrix} \right] \right)} \cdot \sqrt{\frac{1}{12} \left( \left[ \begin{smallmatrix} \text{red} & \text{blue} \\ \text{blue} & \text{blue} \end{smallmatrix} \right] + \left[ \begin{smallmatrix} \text{blue} & \text{blue} \\ \text{green} & \text{blue} \end{smallmatrix} \right] + \left[ \begin{smallmatrix} \text{blue} & \text{blue} \\ \text{blue} & \text{blue} \end{smallmatrix} \right] \right)} = 3 \cdot \sqrt{\frac{1}{12} \left( \left[ \begin{smallmatrix} \text{red} & \text{blue} \\ \text{blue} & \text{blue} \end{smallmatrix} \right] + \left[ \begin{smallmatrix} \text{blue} & \text{blue} \\ \text{green} & \text{blue} \end{smallmatrix} \right] + \left[ \begin{smallmatrix} \text{blue} & \text{blue} \\ \text{blue} & \text{blue} \end{smallmatrix} \right] \right)} \cdot \sqrt{\frac{1}{12} \left( \left[ \begin{smallmatrix} \text{red} & \text{blue} \\ \text{blue} & \text{blue} \end{smallmatrix} \right] + \left[ \begin{smallmatrix} \text{blue} & \text{blue} \\ \text{green} & \text{blue} \end{smallmatrix} \right] + \left[ \begin{smallmatrix} \text{blue} & \text{blue} \\ \text{blue} & \text{blue} \end{smallmatrix} \right] \right)}
 \end{aligned}$$

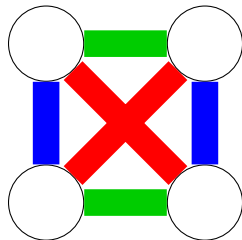
Can be written without flags, a simple counting proof.

THEOREM (CHAO AND HANS YU 2024+)

$$\text{Triangle} \leq 3 \sqrt{\text{Blue} \cdot \text{Green} \cdot \text{Red}}$$


THEOREM (BALOGH, BRADSHAW, GARCIA, L. 2025+)

$$\text{Square} \leq \frac{3}{2} \cdot \left( \text{Blue} \times \text{Green} \times \text{Red} \right)^{2/3}$$




We also have exactness and translation to counting and a short entropy proofs.



## Flag Algebras and Its Application

## └ Flag Algebras

THEOREM (CHAO AND HANS YU 2024+)

$$\nabla \leq 3 \sqrt{\frac{1}{2} \left( \frac{1}{2} \right)}$$

THEOREM (BALOGH, BRAIDSHAW, GARCIA, L. 2025+)

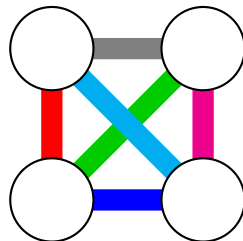
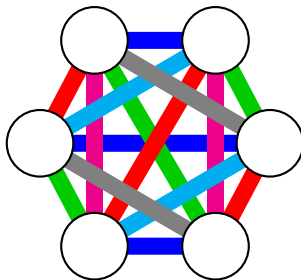
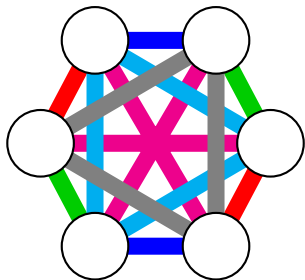
$$\frac{1}{2} \left( \frac{1}{2} \right) \leq \left( \frac{1}{2} \right)^{2/3}$$

We also have exactness and translation to counting and a short entropy proofs.



We also have exactness results.

## FURTHER DIRECTIONS



### QUESTION

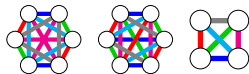
Let  $G$  be a graph with edges colored by colors  $\{1, \dots, 6\}$ . Denote by  $C_i$  the number of edges colored by color  $i$ . Let  $H$  be the number of rainbow copies of  $K_4$  in  $G$ . Is it true that  $H \leq \sqrt[3]{\prod_i C_i}$ ?

## Flag Algebras and Its Application

└ Flag Algebras

└ Further Directions

## FURTHER DIRECTIONS



## QUESTION

Let  $G$  be a graph with edges colored by colors  $\{1, \dots, 6\}$ . Denote by  $C_i$  the number of edges colored by color  $i$ . Let  $H$  be the number of rainbow copies of  $K_4$  in  $G$ . Is it true that  $H \leq \sqrt[4]{\prod_{i=1}^6 C_i}$ ?

We note how to do it for a fixed rainbow coloring

# Mathematical Biology

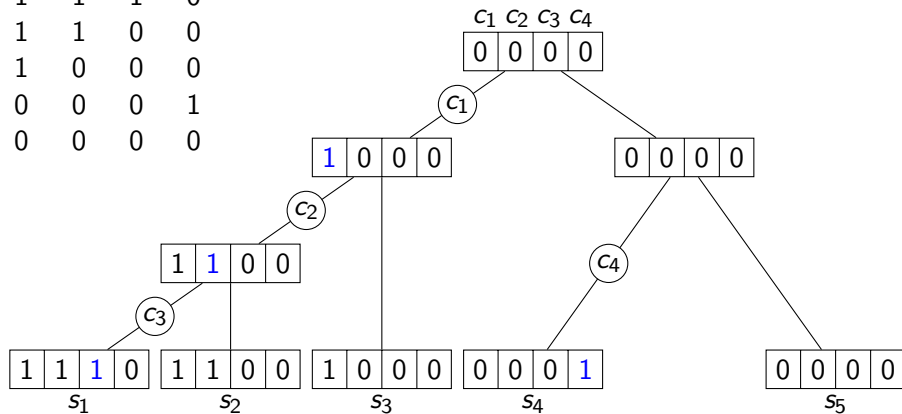
# PHYLOGENY

How do we reconstruct an evolutionary history (*phylogeny*) from observations of living species from characters?

	Spine	Fur	Fins	Wings
Seal	1	1	1	0
Dog	1	1	0	0
Lizard	1	0	0	0
Butterfly	0	0	0	1
Worm	0	0	0	0

# PHYLOGENY

	$c_1$	$c_2$	$c_3$	$c_4$
$s_1$	1	1	1	0
$s_2$	1	1	0	0
$s_3$	1	0	0	0
$s_4$	0	0	0	1
$s_5$	0	0	0	0



# ASSUMPTIONS AND PROBLEMS

*Perfect phylogeny model* is fundamental, but inaccurate

LEMMA (THREE-GAMETE CONDITION, HUDSON-KAPLAN '85)

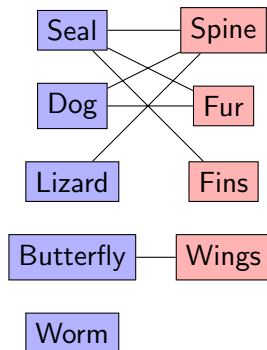
*A set of species and characters has a perfect phylogeny if and only if for every pair of traits, no three species present all of the combinations 10, 01, 11.*

	Spine	Fins
Seal	1	1
Lizard	1	0
Butterfly	0	0

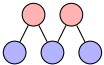
	Spine	Wings
Bird	1	1
Lizard	1	0
Butterfly	0	1

# INCIDENCE GRAPH

	Spine	Fur	Fins	Wings
Seal	1	1	1	0
Dog	1	1	0	0
Lizard	1	0	0	0
Butterfly	0	0	0	1
Worm	0	0	0	0



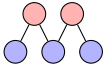
## CLAIM

*The incidence graph contains no induced copies of  if and only if the three-gamete condition is satisfied*



# MAIN PROBLEM

## PROBLEM

How many induced copies of  $M =$   can we possibly have?

Counting  $M$  measures how far from perfect.

- Inducibility problem in *red-blue graphs*
- Bipartite graphs with fixed two-colorings
- Isomorphisms are graph isomorphisms and preserve colors

# RESULTS

THEOREM (EULENSTEIN, HALFPAP, L., MIYASAKI, PFENDER, VOLEC 2025+)

Fix  $\alpha > 0$ . Let  $G_{r,b}$  be a red-blue graph with  $r$  red vertices and  $b$  blue vertices with  $\frac{r}{b} = \alpha$ . Then

$$\# \begin{array}{c} \text{red} \quad \text{red} \\ \diagdown \quad \diagup \\ \text{blue} \quad \text{blue} \quad \text{blue} \end{array} \leq \frac{r^2 b^3}{81} + o(r^2 b^3)$$

COROLLARY (EHLMPV)

If  $G_n$  is a red-blue graph on  $n$  vertices then

$$\# \begin{array}{c} \text{red} \quad \text{red} \\ \diagdown \quad \diagup \\ \text{blue} \quad \text{blue} \quad \text{blue} \end{array} \leq \frac{2^2 3^4 n^5}{15^5} + o(n^5).$$

# ASYMPTOTIC EXTREMAL EXAMPLES

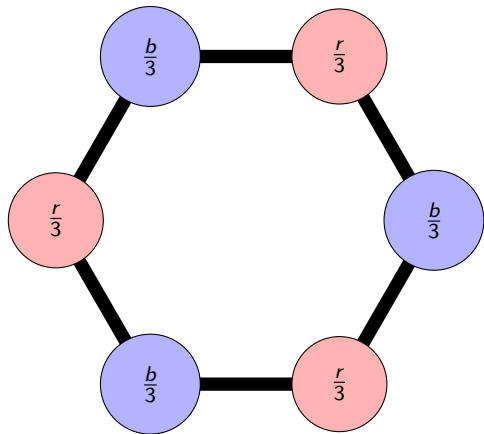


FIGURE:  $C_6(r, b)$

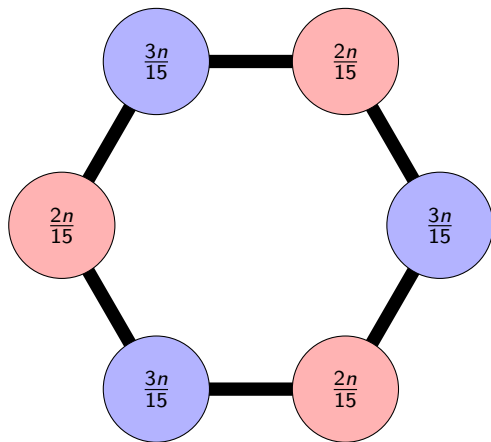
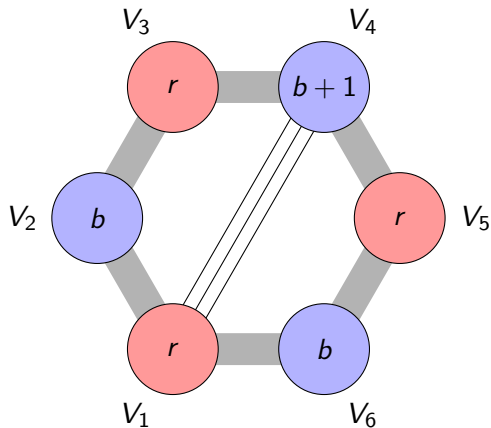


FIGURE:  $C_6(n)$

# FUTURE DIRECTIONS

- Determine lower-order terms and stronger characterizations



## $\varepsilon$ -similar Triangles

2025-12-12

Flag Algebras and Its Application

└ Flag Algebras

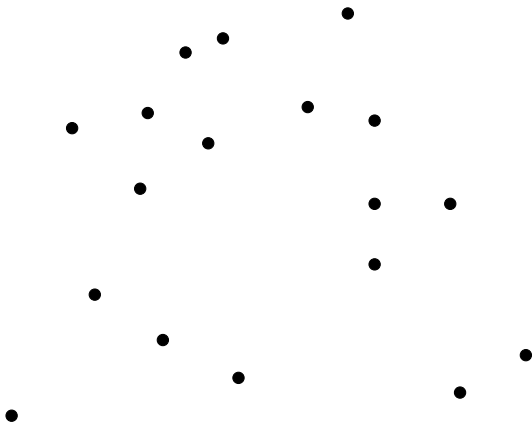
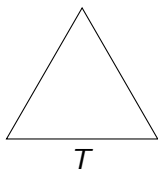
$\varepsilon$ -similar Triangles

<https://arxiv.org/abs/2101.10304>

## PROBLEM

Let  $T$  be a triangle and  $n \in \mathbb{N}$  fixed.

Which  $n$  points in  $\mathbb{R}^2$  maximize the number of triangles similar to  $T$ ?

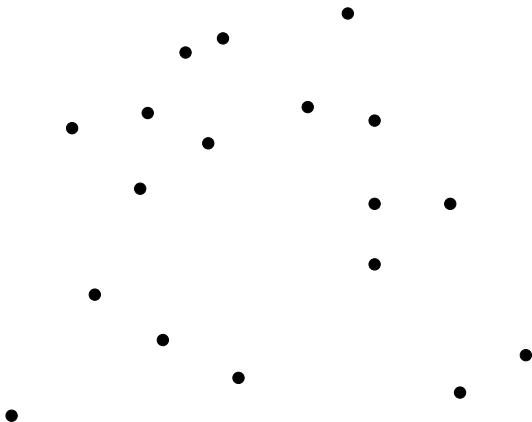
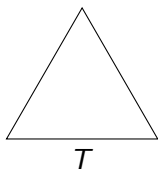


$T_1$  and  $T_2$  are  *$\varepsilon$ -similar* if their inner angles differ by at most  $\varepsilon$ .  
(OK to move, scale, rotate,  $\varepsilon$ -perturb)

## PROBLEM

Let  $T$  be a triangle and  $n \in \mathbb{N}$  fixed. (and  $\varepsilon > 0$  fixed)

Which  $n$  points in  $\mathbb{R}^2$  maximize the number of triangles similar to  $T$ ?



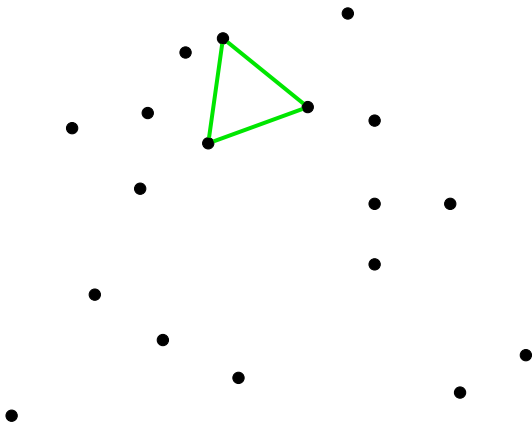
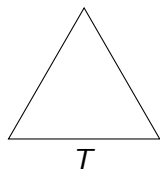
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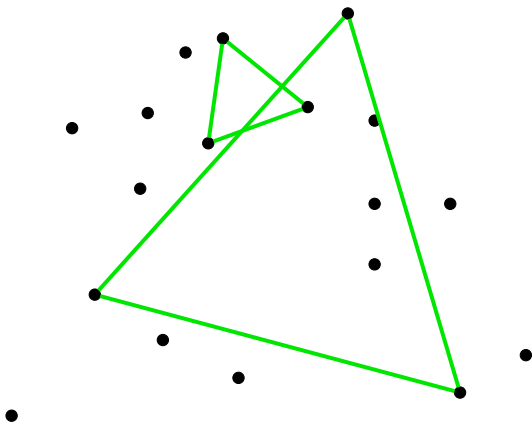
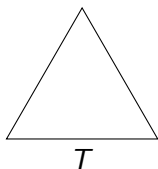


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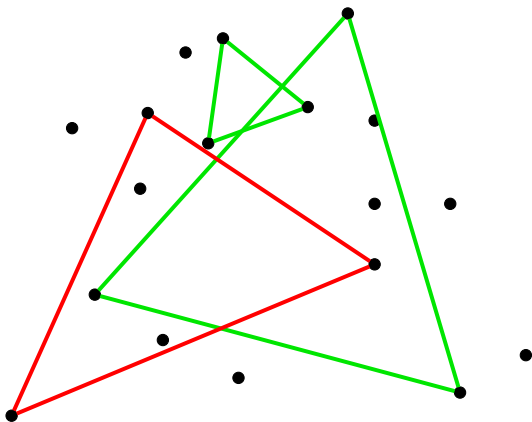
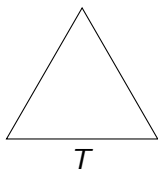


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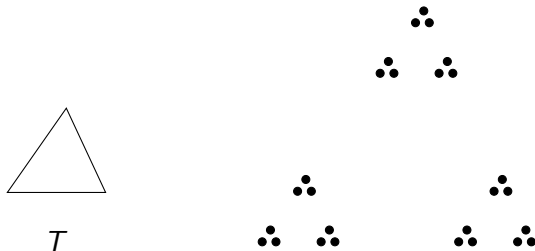


$T_1$  and  $T_2$  are  *$\varepsilon$ -similar* if their inner angles differ by at most  $\varepsilon$ .  
(OK to move, scale, rotate,  $\varepsilon$ -perturb)

## LOWER BOUND CONSTRUCTION

Let  $T$  be a triangle and  $n \in \mathbb{N}$  fixed. (and  $\varepsilon > 0$  fixed)

Which  $n$  points in  $\mathbb{R}^2$  maximize the number of triangles similar to  $T$ ?



$h(n, T, \varepsilon) := \max \#$  of  $\varepsilon$ -similar triangles to  $T$ , it is at least  $\frac{1}{4} \binom{n}{3} (1 + o(1))$ .

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$T$



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# RESULTS

## THEOREM (BÁRÁNY AND FÜREDI (2019))

*For almost every triangle  $T$  there is an  $\varepsilon_0 > 0$  such that for all  $0 < \varepsilon \leq \varepsilon_0$ ,*

$$h(n, T, \varepsilon) \leq 0.25072 \binom{n}{3} (1 + o(1)).$$

*If  $T$  is equilateral, then  $h(n, T, \varepsilon) = \frac{1}{4} \binom{n}{3} (1 + o(1))$*

## THEOREM (BALOGH, CLEMEN, L. (2022))

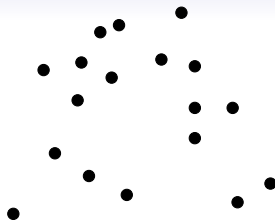
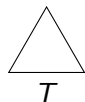
*For almost every triangle  $T$  there is an  $\varepsilon_0 > 0$  such that for all  $0 < \varepsilon \leq \varepsilon_0$ ,*

$$h(n, T, \varepsilon) = \frac{1}{4} \binom{n}{3} (1 + o(1)).$$

$h(n, T, \varepsilon) := \max \#$  of  $\varepsilon$ -similar triangles to  $T$ , it is at least  $\frac{1}{4} \binom{n}{3} (1 + o(1))$ .

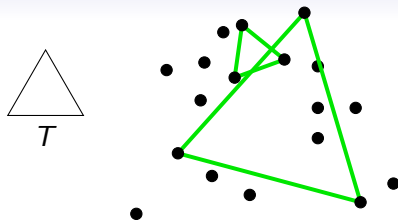
Let  $T$  and  $\varepsilon$  are given

- Fix  $n$  points in the plane.
- For every  $T'$   $\varepsilon$ -similar to  $T$ , add a 3-edge
- Investigate the resulting hypergraph  $H$   
 $H$  has no subhypergraph in  $\mathcal{F} = \{K_4^3, \dots\}$



Let  $T$  and  $\varepsilon$  are given

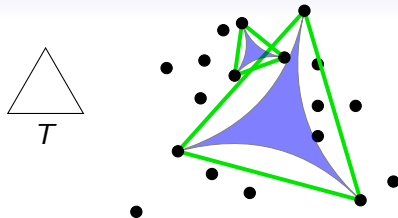
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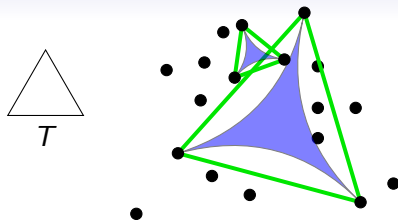


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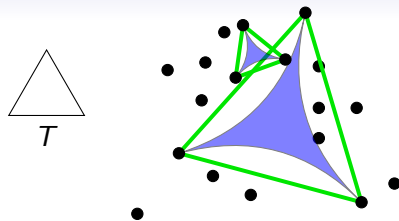
THEOREM (BALOGH, CLEMEN, L. (2022))

$\mathcal{F}$ -free hypergraph has at most  $\frac{1}{4} \binom{n}{3} (1 + o(1))$  edges.



Let  $T$  and  $\varepsilon$  are given

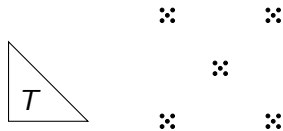
- Fix  $n$  points in the plane.
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 $H$  has no subhypergraph in  $\mathcal{F} = \{K_4^3, \dots\}$



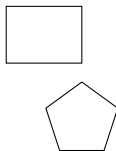
**THEOREM (BALOGH, CLEMEN, L. (2022))**

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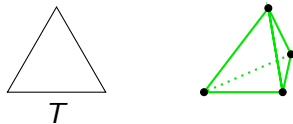
All triangles?



Other Shapes?



in  $\mathbb{R}^d$ ?



## Counting $k$ -SAT functions

## QUESTION

Count functions

$$f : \{0, 1\}^n \rightarrow \{0, 1\} \quad 2^{2^n}$$

*k-SAT FUNCTION* can be defined as

$$f(x_1, \dots, x_n) = C_1 \vee C_2 \vee \dots \vee C_m$$

$$C_i = \underbrace{z_1 \wedge z_2 \wedge \dots \wedge z_k}_{\text{all different variables}} \quad z_i \in \{x_1, \neg x_1, x_2, \neg x_2, \dots, x_n, \neg x_n\}$$

$x_i$  variable,  $C_i$  clause,  $z_i$  literal

example  $k = 3$

$$x_1 \wedge x_2 \rightarrow (x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge \neg x_3)$$

$$x_1 \wedge x_2 \wedge \neg x_2 \rightarrow \text{always false}$$

Every  $k$ -SAT function has a formula but the formula may not be unique.

number of  $f : \{0, 1\}^n \rightarrow \{0, 1\}$   $2^{2^n}$

number of  $k$ -SAT formula  $2^{2^k \binom{n}{k}}$

number of  $k$ -SAT functions?

$k$ -SAT formula is *monotone* if it uses only  $x_1, x_2, \dots, x_n$ , (i.e. no  $\neg x_i$  is used)

All monotone  $k$ -SAT formula give different functions

$$g \notin x_1 \wedge \dots \wedge x_k \ni f \quad f \neq g \text{ at } x_1 = \dots = x_k = 1, x_{k+1} = \dots = x_m = 0$$

Number of monotone  $k$ -SAT functions  $2^{\binom{n}{k}}$

$k$ -SAT formula is *unate* if it uses at most one of  $\{x_i, \neg x_i\} = \{x_i, \overline{x_i}\}$

Number of unate  $k$ -SAT functions  $(1 + o(1))2^{n + \binom{n}{k}}$

Functions avoiding  $x_i$  counted multiple times

## CONJECTURE (BOLLOBÁS, BRIGHTWELL, LEADER 2003)

Fix  $k \geq 2$ ,  $1 - o(1)$  fraction of  $k$ -SAT functions are unate as  $n \rightarrow \infty$ .  $(1 + o(1))2^{n + \binom{n}{k}}$

- # 2-SAT functions is  $2^{(1+o(1))\binom{n}{2}}$ . Bollobás, Brightwell, Leader 2003  
using Szemerédi regularity lemma
- Conjecture true for  $k = 2$  Allen 2007  
using Szemerédi regularity lemma
- Conjecture true for  $k = 2$  Ilinca, Kahn 2009  
without Szemerédi regularity lemma
- Conjecture true for  $k = 3$  Ilinca, Kahn 2012  
using hypergraph regularity lemma
- Conjecture true for  $k = 4, 5$  Dong, Mani, Zhao 2022

Conjecture true for all  $k$  :-) Balogh, Dong, Lidický, Mani, Zhao

- Step 1: Reduction to a Turán type problem  
*Dong, Mani, Zhao using blow-up, saturation, container method*
- Step 2: Solving the extremal problem  
*Balogh, Dong, L., Many, Zhao: computer free flag-algebra*

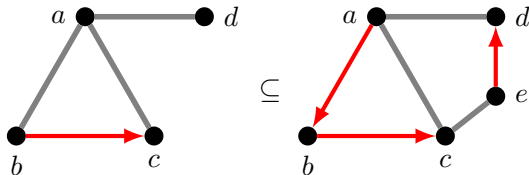


# DIRECTED HYPERGRAPH TURÁN PROBLEM

*Partially directed  $k$ -graph* is a  $k$ -uniform hypergraph, where every edge is

- undirected
- rooted at one vertex (directed towards one vertex)

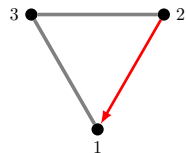
$\vec{H} \subseteq \vec{G}$  if  $\vec{H}$  could be obtained from  $\vec{G}$  by removing some vertices, edges, or orientations.



$\vec{T}_k$ 

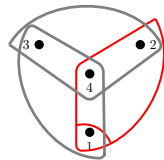
- $\vec{T}_2 = \{\hat{1}2, 13, 23\}$ 

①	②	③
-1-	-2-	...
-1-	...	-3-
...	-2-	-3-



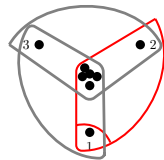
- $\vec{T}_3 = \{\hat{1}24, 134, 234\}$ 

①	②	③	④
-1-	-2-	...	-4-
-1-	...	-3-	-4-
...	-2-	-3-	-4-



- $\vec{T}_k = \{\hat{1}24 \cdots k+1, 134 \cdots k+1, 234 \cdots k+1\}$

①	②	③	④	⑤
-1-	-2-	...	-4-	-5-
-1-	...	-3-	-4-	-5-
...	-2-	-3-	-4-	-5-



# EXTREMAL PROBLEM

$G$  is  $k$ -uniform,  $n$ -vertex,  $\vec{T}_k$ -free.



$$\alpha := \frac{e_{\text{undirected}}(G)}{\binom{n}{k}}$$



$$\beta := \frac{e_{\text{directed}}(G)}{\binom{n}{k}}$$

Given  $k, \theta$ , what is

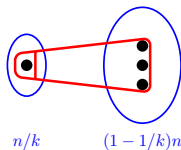
$$\max\{\alpha + \theta\beta\}?$$

Special (open) case:

Show  $\alpha + \theta\beta \leq 1$  when  $1 \leq \theta \leq \left(1 - \frac{1}{k}\right)^{1-k} \approx e$

Constructions:

Complete undirected graph



CONJECTURE (BOLLOBÁS, BRIGHTWELL, LEADER 2003)

*Fix  $k \geq 2$ , almost all  $k$ -SAT functions are unate.*

THEOREM (DONG, MANI, ZHAO)

*If  $\alpha + \theta\beta \leq 1$  for some  $\theta > \log_2 3$  then almost all  $k$ -SAT functions are unate.*

This theorem is a lot of work. Uses hypergraph containers by Balogh, Morris, Samotij; Saxton, Thomasson (AMS Steele prize 2024)

THEOREM (DONG, MANI, ZHAO)

*Conjecture true for  $k \leq 5$ .*

THEOREM (BALOGH, DONG, LIDICKÝ, MANI, ZHAO)

*Conjecture true for all  $k$ .*

# PROOF FOR $k \geq 4$ USING FLAG ALGEBRAS

If graphs represent densities as

$$\begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \bullet \\ \hline \text{--}1\text{--} \text{--}2\text{--} \text{--}3\text{--} \text{--}4\text{--} \end{array} = \alpha := \frac{e_{undirected}(G)}{\binom{n}{k}}$$

$$\begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \bullet \\ \hline \text{--}\textcolor{red}{1}\text{--} \text{--}2\text{--} \text{--}3\text{--} \text{--}4\text{--} \end{array} = \beta := \frac{e_{directed}(G)}{\binom{n}{k}}$$

then

$$\begin{aligned} & \alpha + \theta\beta \\ = & \begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \bullet \\ \hline \text{--}1\text{--} \text{--}2\text{--} \text{--}3\text{--} \text{--}4\text{--} \end{array} + \theta \begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \bullet \\ \hline \text{--}\textcolor{red}{1}\text{--} \text{--}2\text{--} \text{--}3\text{--} \text{--}4\text{--} \end{array} \\ \leq & \begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \bullet \\ \hline \text{--}1\text{--} \text{--}2\text{--} \text{--}3\text{--} \text{--}4\text{--} \end{array} + \theta \begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \bullet \\ \hline \text{--}\textcolor{red}{1}\text{--} \text{--}2\text{--} \text{--}3\text{--} \text{--}4\text{--} \end{array} + \left[ \left( a \begin{array}{|c|} \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline \end{array} \begin{array}{|c|} \hline 3 \\ \hline \end{array} \bullet - b \begin{array}{|c|} \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline \end{array} \begin{array}{|c|} \hline 3 \\ \hline \end{array} \textcolor{red}{\bullet} \begin{array}{|c|} \hline 4 \\ \hline \end{array} \right)^2 \right] \\ \leq & 1 \end{aligned}$$

$$\text{for } \theta = 1 + \frac{1}{\sqrt{2}} \geq 1.707 > \log_2 3 \quad a = \frac{1}{\sqrt{2}}, b = \frac{k(\theta-1)-1}{\sqrt{2}} \quad k \geq 4$$

# PROOF FOR $k = 2$ AND $k = 3$

$$1 \begin{array}{c} \bullet \bullet \\ -1- -2- \end{array} + 1.7 \begin{array}{c} \bullet \bullet \\ -1- -2- \end{array} + \left[ \left( -1 \begin{array}{c} \boxed{1} \bullet \\ -1- -2- \end{array} - 1 \begin{array}{c} \boxed{1} \bullet \\ -1- -2- \end{array} + 0.98 \begin{array}{c} \boxed{1} \bullet \end{array} \right)^2 \right] \leq 1$$

$$1 \begin{array}{c} \bullet \bullet \bullet \\ -1- -2- -3- \end{array} + 1.7 \begin{array}{c} \bullet \bullet \bullet \\ -1- -2- -3- \end{array} + 0.039 \times \left[ \left( -6 \begin{array}{c} \boxed{1} \boxed{2} \bullet \\ -1- -2- -3- \end{array} - 5 \begin{array}{c} \boxed{1} \boxed{2} \bullet \\ -1- -2- -3- \end{array} + 5 \begin{array}{c} \boxed{1} \boxed{2} \bullet \end{array} \right)^2 \right] \leq 1$$

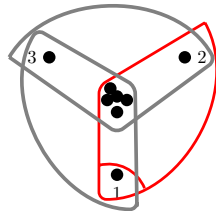
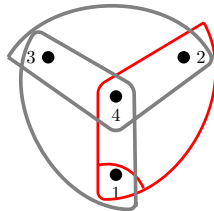
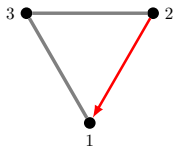
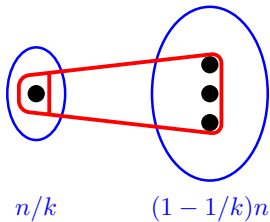
# FUTURE DIRECTIONS

THEOREM (BALOGH, DONG, LIDICKÝ, MANI, ZHAO)

If  $\vec{T}_k$  is forbidden, then  $\bullet_{-1} \bullet_{-2} \bullet_{-3} \bullet_{-4} + \left(1 + \frac{1}{\sqrt{2}}\right) \bullet_{-1} \bullet_{-2} \bullet_{-3} \bullet_{-4} \leq 1$  for all  $k$ .

QUESTION

If  $\vec{T}_k$  is forbidden, then  $\bullet_{-1} \bullet_{-2} \bullet_{-3} \bullet_{-4} + \left(1 - \frac{1}{k}\right)^{1-k} \bullet_{-1} \bullet_{-2} \bullet_{-3} \bullet_{-4} \leq 1$  for all  $k$ ?



## **Temporary page!**

$\text{\LaTeX}$  was unable to guess the total number of pages correctly. As there was some unprocessed data that should have been added to the final page this extra page has been added to receive it.

If you rerun the document (without altering it) this surplus page will go away, because  $\text{\LaTeX}$  now knows how many pages to expect for this document.