Positive co-degree densities and jumps

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Extremal Numbers for Graphs

$\text{ex}(n, K_k)$ = maximum number of edges in K_k-free graph

Theorem (Turán 1941)

$$ex(n, K_k) = \frac{k-2}{2(k-1)} n^2 + o(n)$$

$T_2(n)$ $T_3(n)$ $T_4(n)$ $T_5(n)$

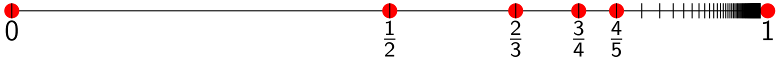
$$\pi(F) = \lim_{n\to\infty} \frac{ex(n,F)}{\binom{n}{2}}$$

Theorem (Erdős-Stone 1946)

$$ex(n,G) = \frac{\chi(G)-2}{2(\chi(G)-1)} n^2 + o(n^2) \qquad \pi(G) = \frac{\chi(G)-2}{\chi(G)-1}$$

$\pi(\mathcal{F}) = min\{\pi(F), F \in \mathcal{F}\}$

$\pi(\mathcal{F}) \in \{\frac{k-1}{k} : k \in \mathbb{N}\}$



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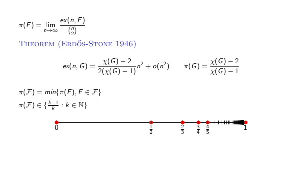
Turán Densities



We know all possible Turán densities even for families of forbidden graphs.

The line is showing possible densities

Extremal Numbers for (3-uniform) hypergraphs

$\pi(\mathcal{F})$ maximum density of edges in $\mathcal{F}$-free hypergraph is difficult

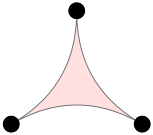
$5/9 \leq \pi(K_4^3) \leq 0.5615 \quad 2/7 \leq \pi(K_4^{3-}) \leq 0.28689$

$\pi(F_5) = 2/9 \quad \pi(F_{3,2}) = 4/9$

$\pi(F_{3,3}) = 3/4 \quad \pi(C_\ell^-) \in \{0, 1/4\}$

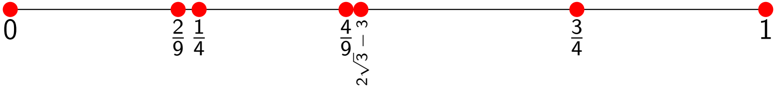
$\pi(C_\ell) \in \{0, 2\sqrt{3} - 3\}$ for large ℓ

No analogue of Erdős-Stone.



Theorem (Balogh 2002)

There exists $\mathcal{F}$ with $\pi(\mathcal{F}) < min\{\pi(F), F \in \mathcal{F}\}$.



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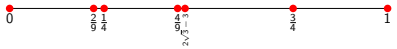


We will only consider 3-uniform hypergraphs, so edges are triples as on the right.

We don't know K_4^3 it is for \$500.

Some of the sporadic results are listed. Dylan King convinced us these are hard to get so here are some of them.

Family is not determined by the minimum so things may get wild.

Hypergraphs jump

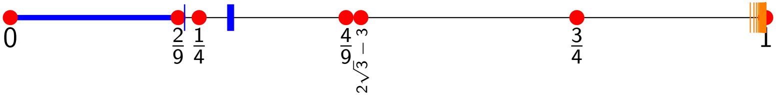
α is an *achievable* if exists $\mathcal{F}$ with $\pi(\mathcal{F}) = \alpha$.

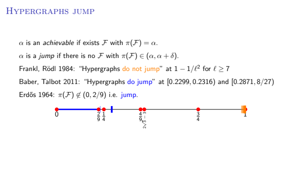
α is a *jump* if there is no $\mathcal{F}$ with $\pi(\mathcal{F}) \in (\alpha, \alpha+\delta)$.

Frankl, Rödl 1984: "Hypergraphs do not jump" at $1 - 1/\ell^2$ for $\ell \geq 7$

Baber, Talbot 2011: "Hypergraphs do jump" at $[0.2299, 0.2316)$ and $[0.2871, 8/27)$

Erdős 1964: $\pi(\mathcal{F}) \notin (0, 2/9)$ i.e. jump.





Say nothing is in $(0, 2/9)$. If $\pi(\mathcal{F})$ is positive, then there is a construction with a positive density of edges. Hence it also contains a blow-up of an edge. So blow-up of an edge is a construction for a lower bound.

$2/9$ is F_5 $4/9$ is $F_3, 2$ $3/4$ is Fano plane. $2\sqrt{3} - 3$ is long cycles

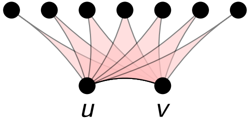
Codegree

codegree$(u, v) := |\{e : u, v \in e \in E\}|$

$\delta_2(G)$ minimum codegree

$coex(n, \mathcal{F}) := \max\{\delta_2(G) : \mathcal{F}\text{-free } n\text{-vertex } G\}$

$$\gamma(\mathcal{F}) := \lim_{n\to\infty} \frac{coex(n, \mathcal{F})}{n}$$



Theorem (Mubayi-Zhao 2007)

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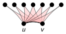


Theorem (Mubayi-Zhao 2007)

γ *does not jump*

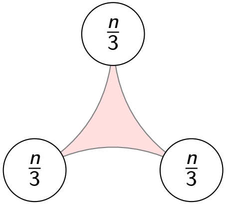
In the picture on the right, codegree of u and v is 7 while the δ_2 is 0 since there are vertices that are in no edge together.

We do it in 3-uniform hypergrahs, it can also be done in k-uniform and then one would take a set of $k-1$ vertices and study the common degree.

The achievable values form a dense set.

Conjecture is that for every $\alpha \in [0,1)$ there is a family with $\gamma(\mathcal{F}) = \alpha$.

Positive co-degree



$\delta_2(G) = 0$
$\delta_2^+(G) = n/3$

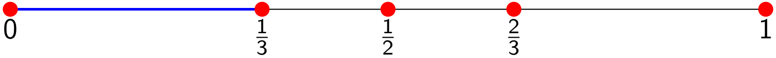
$\delta_2^+(G)$ minimum positive codegree

$co^+ex(n, \mathcal{F}) := \max\{\delta_2^+(G) : \mathcal{F}\text{-free } n\text{-vertex } G\}$

$$\gamma^+(\mathcal{F}) := \lim_{n\to\infty} \frac{co^+ex(n,\mathcal{F})}{n}$$

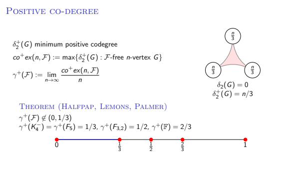
Theorem (Halfpap, Lemons, Palmer)

$\gamma^+(\mathcal{F}) \notin (0, 1/3)$
$\gamma^+(K_4^-) = \gamma^+(F_5) = 1/3$, $\gamma^+(F_{3,2}) = 1/2$, $\gamma^+(\mathbb{F}) = 2/3$



Turán Densities

└ Positive co-degree



Empty graph is defined to have positive codegree zero.

Mike Santana had a note about blow-ups so this definition works well with blow-ups.

Our contribution

Theorem (Halfpap, Lemons, Palmer)

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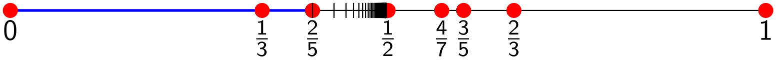
Theorem (Balogh, Halfpap, L., Palmer)

$\gamma^+(\mathcal{F}) \notin (1/3, 2/5)$
$\gamma^+(K_4^3, F_{3,2}, J_k) = (k-2)/(2k-3)$
$\gamma^+(F_1) = 2/5$, $\gamma^+(J_4) = 4/7$, $\gamma^+(F_{4,2}) = 3/5$



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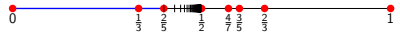

The first Theorem generalizes to r-uniform as $1/r$ and $1/(2r-1)$.

Tools

Theorem (Removal Lemma)

If densities of $\mathcal{F}$ in G are $o(1)$ then removal of $o(n^3)$ edges makes it $\mathcal{F}$-free.

Theorem (Halfpap-Lemons-Palmer)

If G' is obtained from a nice G by removing $o(n^3)$ edges,
G' has a subgraph G'' with $\delta_2^+(G'')$ close to $\delta_2^+(G)$.

Theorem (Halfpap-Lemons-Palmer)

If $\delta_2^+(G) \geq cn$ then $|E(G)| \geq \frac{c^3}{2}\binom{n}{3}$.

Theorem (Halfpap-Lemons-Palmer)

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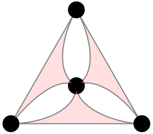
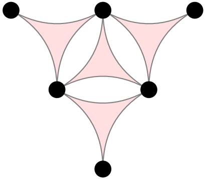
These also works form r-uniform hypergraphs.

$\gamma(\mathcal{F}) \notin (1/3, 2/5)$

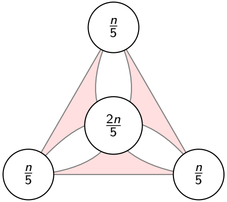
If $\mathcal{F}$ forbids T_3 then $\gamma^+(\mathcal{F}) \leq 1/3$.



T_3



If $\mathcal{F}$ permits T_3, blow-up T_3



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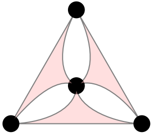

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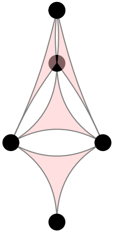

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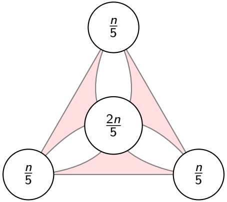

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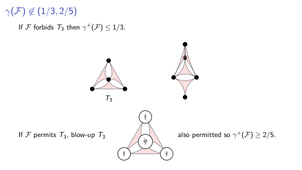
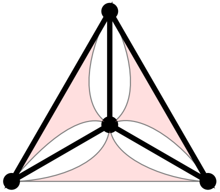


This generalizes to r-uniform hypergraphs but lets not worry about it now.

Suppose for contradiction $\mathcal{F}$ forbids T_3 and positive codegree is more than $1/3$. Since it is positive, there is an edge. All three pairs of vertices of the edge have positive codegree. Since $3 * (1/3 + \varepsilon) > 1$, there must be a vertex in an intersection of two of these, and that gives T_3.

$\gamma^+(K_4^3, F_{3,2}, J_4) = 2/5$

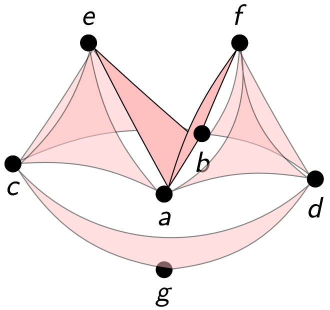
If positive codegree $> 2/5$, find T_3



3 of the pairs have a common neighbor, find K_4^3, $F_{3,2}$ or J_4.

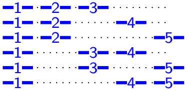
$\gamma^+(F_1) = 2/5$

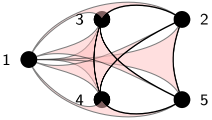
$\gamma^+(K_4^3, F_{3,2}, J_4) = 2/5$
If positive codegree $> 2/5$, find a blow-up of K_4^3 or $F_{3,2}$ or J_4 and then find F_1.

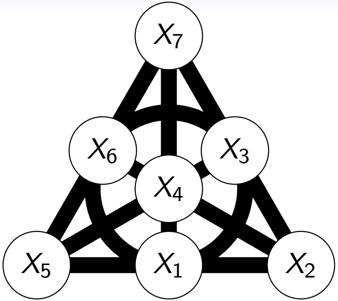


F_1

$\gamma^+(J_4) = 4/7$







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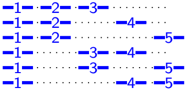
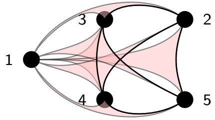
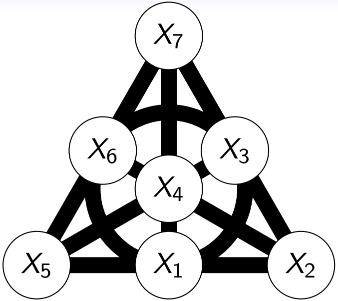
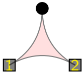
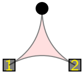
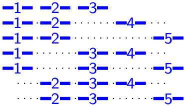
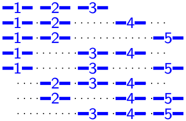


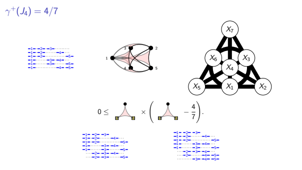




$0 \leq$  $\times \Bigg($  $- \frac{4}{7}\Bigg).$





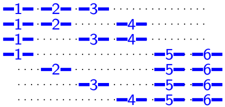


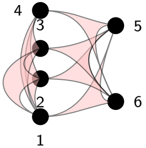
The extremal construction is a COMPLEMENT of the Fano plane on the right. We think the construction is the interesting part.

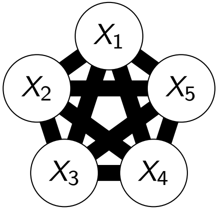
Outline of the proof:

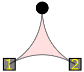
Apply flag algebras. Modeling the positive codegree condition is with the depicted equation. It is saying that if you fix two vertices and count their codegree $/n$, it is either 0 which makes the equation true or at least $4/7$ which again makes the equation true. We get bunch of forbidden structures, using the clean-up lemmas. In particular, there are only two subgraphs on 5 vertices in the construction. Since positive codegree still high, we find K_4. Notice X_1, X_2, X_3, X_4 form a K_4 and each of the 7 vertices is determined by adjacencies to X_1, X_2, X_3, X_4. Hence we can partition the remaining vertices. See how the two graphs allow for either duplicating 4 as the one on the left or missing a matching. The graph on the right is missing $1, 4, 5$ and $2, 3, 5$ so it would place the vertex 5 in X_5. We finish with a little clean-up to get the final structure.

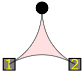
$\gamma^+(F_{4,2}) = 3/5$







$0 \leq$ 

$\times \Bigg($ 

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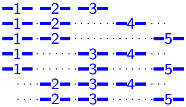
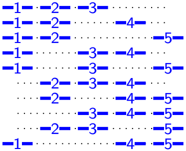
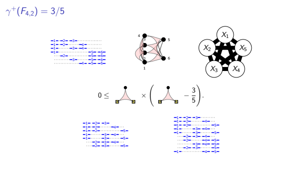






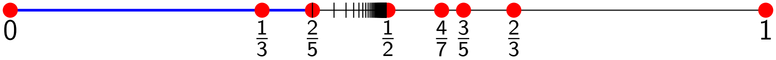
The idea of the proof is the same. The main difference is that the extremal construction is a blow-up of K_5^3 which is not drawn again in a complement but the complement are not 3-edges but 2-edges. Razborov's trick.

Notice that $F_{4,2}$ has two the two remaining but the 4 vertices induce K_4^{3-}.

And you can again see the two graphs on 5 vertices one is a duplicate and the other is K_5^3. It has 10 edges to it has to be, right?

Question

Find admissible values of γ^+ *in* $[\frac{2}{5}, \frac{1}{2}]$ *that are not* $\frac{k-2}{2k-3}$.
Find more jumps for γ^+.
Find not jumps for γ^+.



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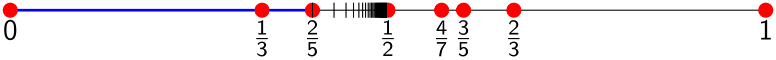
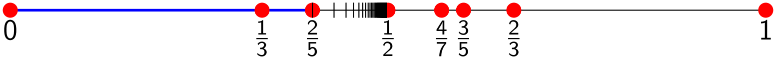


"If γ makes you sad, your life may be more positive with γ^+"

Questions

Question

Find admissible values of γ^+ in $[\frac{2}{5}, \frac{1}{2}]$ that are not $\frac{k-2}{2k-3}$.
Find more jumps for γ^+.
Find not jumps for γ^+.



"If γ makes you sad, your life may be more positive with γ^+"

Thank you!

Best-known density bounds for π, γ, and γ^+.

F	$\leq \pi(F)$	$\pi(F) \leq$	$\leq \gamma(F)$	$\gamma(F) \leq$	$\leq \gamma^+(F)$	$\gamma^+(F) \leq$
K_4^{3-}	$2/7$	0.28689	$1/4$	$1/4$	$1/3$	$1/3$
F_5	$2/9$	$2/9$	0	0	$1/3$	$1/3$
$F_{3,2}$	$4/9$	$4/9$	$1/3$	$1/3$	$1/2$	$1/2$
$\mathbb{F}$	$3/4$	$3/4$	$1/2$	$1/2$	$2/3$	$2/3$
K_4^3	$5/9$	0.5615	$1/2$	0.529	$1/2$	0.543
$F_{3,3}$	$3/4$	$3/4$	$1/2$	0.604	$3/5$	0.616
C_5	$2\sqrt{3}-3$	0.46829	$1/3$	0.3993	$1/2$	$1/2$
C_7	$2\sqrt{3}-3$	0.464186	$1/3$	0.371	$1/2$	$1/2$
C_5^-	$1/4$	$1/4$	0	0	$1/3$	$1/3$
J_4	$1/2$	0.50409	$1/4$	0.473	$4/7$	$4/7$
$F_{4,2}$	$4/9$	0.4933328	$1/3$	0.4185	$3/5$	$3/5$

Not exhaustive table. See our paper for citations and definitions.