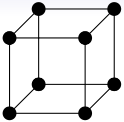
Density of rainbow triangles and properly colored K_4s

József Balogh   Peter Bradshaw   Ramon I. García   Bernard Lidický

AMS Sectional Meeting
Boise, Idaho
Mar 7, 2027

Joints

Joint in $\mathbb{R}^d$ is a point where d lines that span $\mathbb{R}^d$ intersect.
What is the maximum number of joints for N lines?



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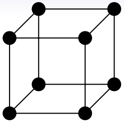

Theorem (Chao and Hans Yu 2023+)

Number of joints is maximized by k hyperplanes whose intersection give $N = \binom{k}{d-1}$ lines and $\binom{k}{d}$ joints.

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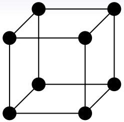

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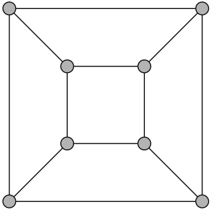
problem: In $\mathbb{R}^3$, maximize joints.

hyperplane $\to$ vertex
intersection of hyperplanes $\to$ edge
joint $\to$ triangle



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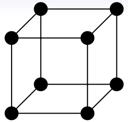

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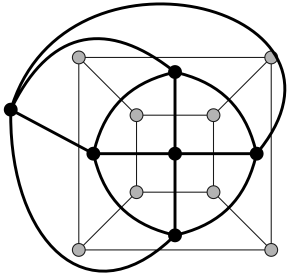

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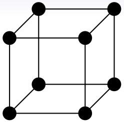

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Multijoint problem: In $\mathbb{R}^3$, lines of three colors, maximize rainbow joints.
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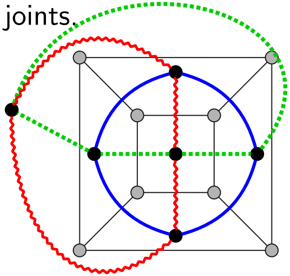
intersection of hyperplanes $\to$ edge
joint $\to$ rainbow triangle



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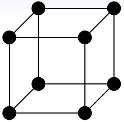

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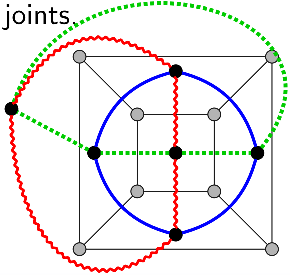
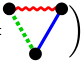


Theorem (Chao and Hans Yu 2024+)

In 3-edge colored graph $\left(\#\right.$  $\left.\right)^2 \leq 2 \left(\#\right.$  $\left.\right) \cdot \left(\#\right.$  $\left.\right) \cdot \left(\#\right.$  $\left.\right)$.

Theorem (Chao and Hans Yu 2024+)

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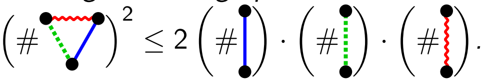

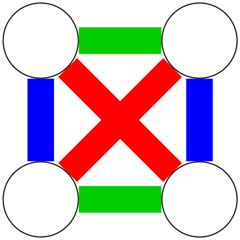

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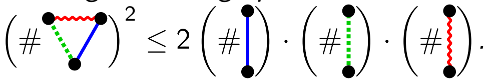

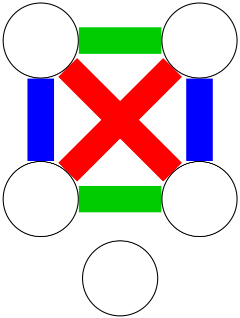

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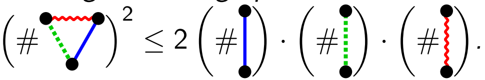
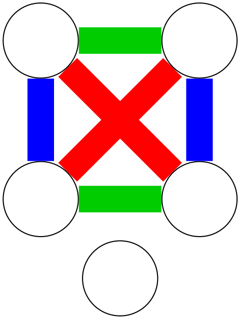
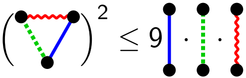


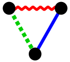


In flag algebras



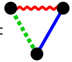
Automated sum-of-squares proof needs 540GB RAM

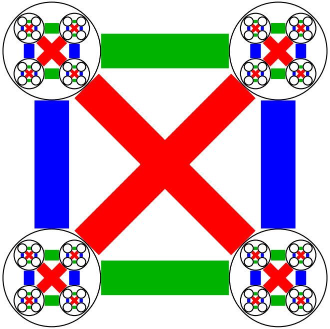
Conjecture (Erdős, Sós 1972-)

3-edge-colored graphs on n vertices maximizing  *are* $\rightarrow$

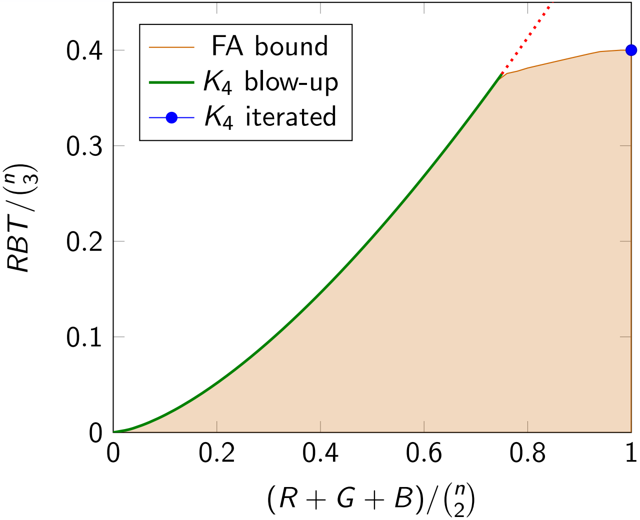
Theorem (Balogh, Hu, L., Pfender, Volec, Young 2017)

Conjecture is true for $n = 4^k$.

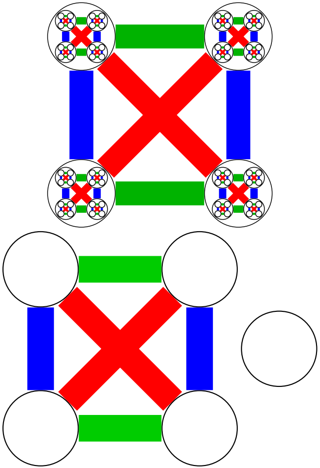
\#  $\leq 0.4\binom{n}{4} + o(n^4)$



PROFILE $R = G = B$



FA bound
K_4 blow-up
K_4 iterated
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1
$RBT/\binom{n}{3}$
$(R+G+B)/\binom{n}{2}$



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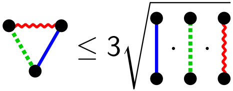
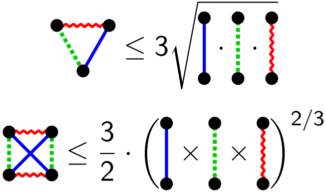
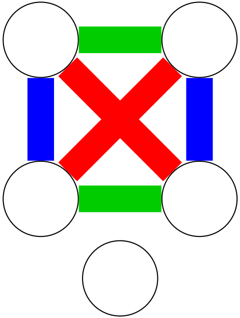
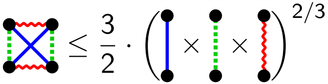


Theorem (Balogh, Bradshaw, García, L.)





We also have exactness, a counting and an entropy proofs.

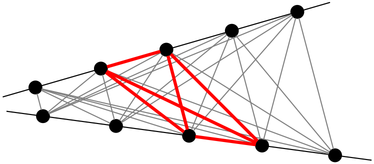


Problem

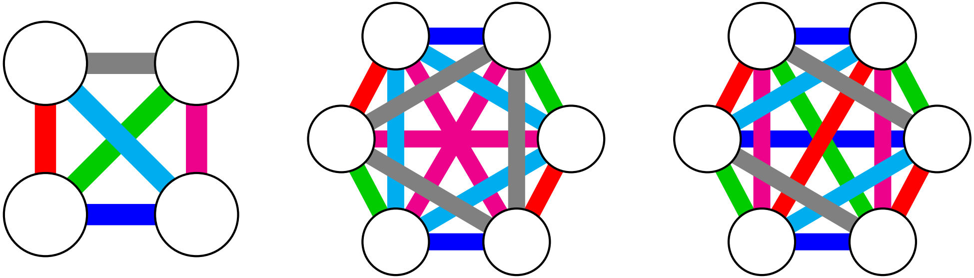
Given N lines in R^3, what is the maximum number tetrahedra that they determine?

k planes give $N = \binom{k}{2}$ lines, which generate $\binom{k}{4} = (\frac{1}{6} + o(1))N^2$ tetrahedra.

2 skew lines, n points on each, $K_{n,n}$ between N lines and $\frac{1}{4} + o(1))N^2$ tetrahedra.

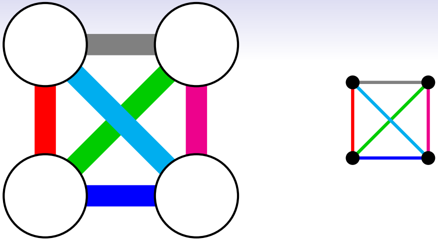


Rainbow cliques



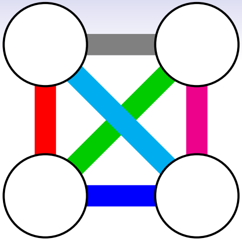
Question

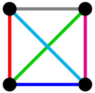
Let G be a graph with edges colored by colors $\{1, \ldots, 6\}$. Denote by C_i the number of edges colored by color i. Let H be the number of rainbow copies of K_4 in G.
Is it true that $H \leq \sqrt[3]{\prod_i C_i}$?



Theorem (Balogh, Bradshaw, García, L. 2025+)

Let G be a graph with edges colored by colors $\{1, \ldots, 6\}$. Denote by C_i the number of edges colored by color i. Let H be the number of **fixed** *rainbow copies of K_4 in G. Then $H \leq \sqrt[3]{\prod_i C_i}$?*





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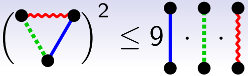
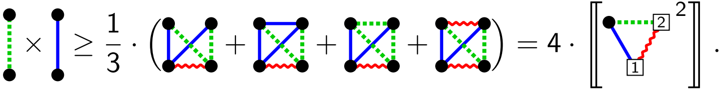
Theorem (Balogh, Bradshaw, García, L. 2026+)

Let $k \geq 4$. Let G be a graph with edges colored by colors $\{1, \dots, \binom{k}{2}\}$. Denote by C_i the number of edges colored by color i. Let H be the number of **fixed** *rainbow copies of K_k in G. Then $H^{k-1} \leq \prod_i C_i$.*

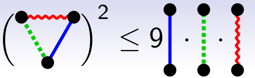
Balogh, Bradshaw, García, L. 2025+

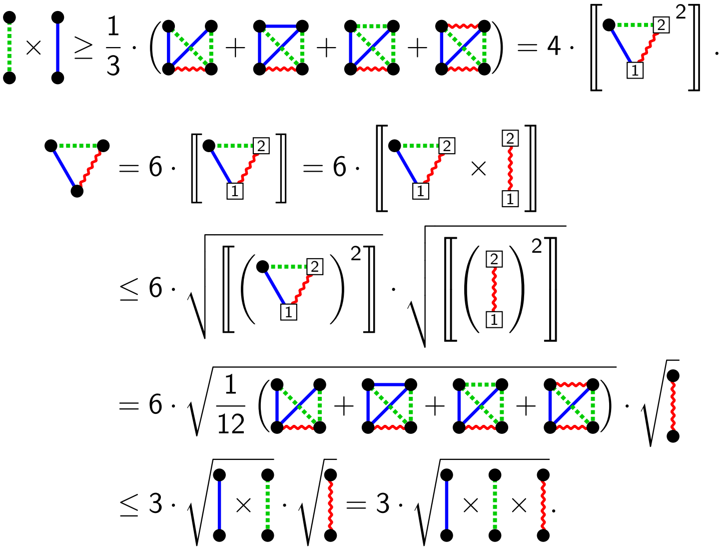




Figs are densities

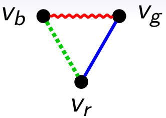
Balogh, Bradshaw, García, L. 2025+

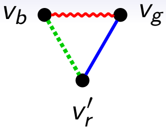




Figs are densities

Entropy proof $H(X) = -\sum_x P(X=x)\log_2(P(X=x))$

Pick randomly 

and resample v_r 

$T \leq \sqrt{2RGB}$

$H(v_r'|v_g, v_b) = H(v_r|v_g, v_b)$

Figs are counts

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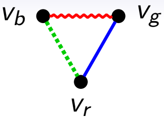
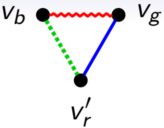
Pick randomly 

and resample v_r 

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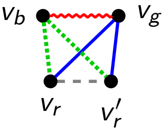
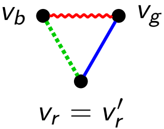
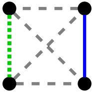
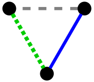
$\# v_r, v_r', v_g, v_b =$ 

$+$ 

$\leq$  $+$  $\leq$  $\times$ 

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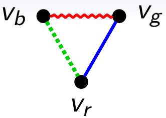
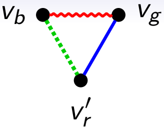
Pick randomly 

and resample v_r 

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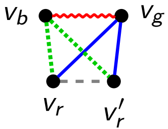
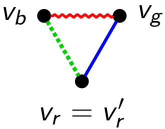
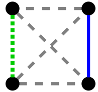
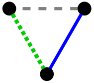
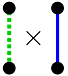
$\#v_r, v_r', v_g, v_b =$ 

$+$ 

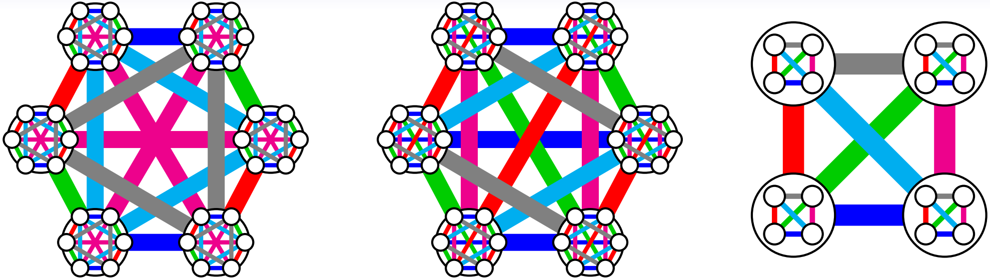
$\le$  $+$  $\le$ 

$$2H(v_r, v_g, v_b) = H(v_r|v_g, v_b) + H(v_g, v_b) + H(v_r'|v_g, v_b) + H(v_g, v_b)$$
$$= H(v_r, v_r'|v_g, v_b) + 2H(v_g, v_b) = H(v_r, v_r', v_g, v_b) + H(v_g, v_b)$$

$$\log_2 T = H(v_r, v_g, v_b) \le \frac{1}{2}H(v_r, v_r', v_g, v_b) + \frac{1}{2}H(v_g, v_b)$$
$$\le \frac{1}{2}(\log_2(G) + \log_2(B) + \log_2(2R))$$

Figs are counts

Rainbow cliques



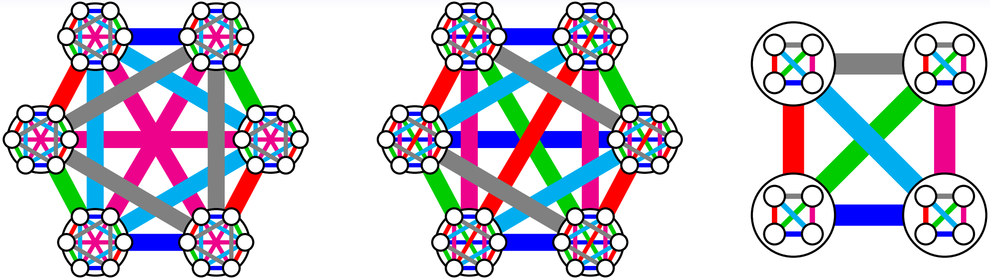
Question

Let G be a graph with edges colored by colors $\{1, \ldots, 6\}$. Denote by C_i the number of edges colored by color i. Let H be the number of rainbow copies of K_4 in G. Are the iterated blow-ups of K_6 above the maximizers?

K_6 blow up gives $\approx \frac{24}{215}\binom{n}{4}$

K_4 blow up gives $\approx \frac{24}{252}\binom{n}{4}$

Rainbow cliques



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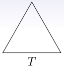
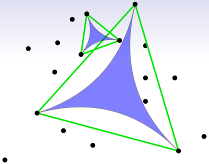
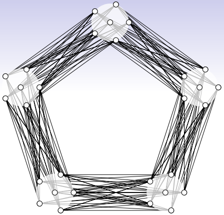
max fixed rainbow copies of K_ℓ for $\ell \geq 11$ solved by Cairncross, Mizgerd and Mubayi.







Thank you for your attention!